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Pedagogical Content Knowledge Within "Mathematical Knowledge for Teaching"

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Pedagogic content knowledge · Mathematical knowledge for teaching · Lee Shulman · Deborah Ball · COACTIV

Definition

Pedagogic content knowledge, in Shulman's (1986, p. 7) terms, refers to: "the most powerful analogies, illustrations, examples, explanations, and demonstrations — [. . .] the most useful ways of representing and formulating the subject that make it comprehensible to others."

Characteristics

Intense focus on the notion of "pedagogical content knowledge" (PCK) within teacher education is attributed to Lee Shulman's 1985 AERA Presidential address (Shulman 1986) in which he referred to PCK as the "special amalgam of content and pedagogy" central to the teaching of subject matter. His widely cited follow-up paper (Shulman 1987) elaborated PCK as follows:

> the most powerful analogies, illustrations, examples, explanations, and demonstrations — [. . .] the most useful ways of representing and formulating the subject that make it comprehensible to others. . .. Pedagogical content knowledge also includes an understanding of what makes the learning of specific topics easy or difficult: the conceptions and preconceptions that students of different ages and backgrounds bring with them. . .. (p. 7)

> the particular form of content knowledge that embodies the aspects of content most germane to its teachability. (p. 9)

Immediate and widespread interest in the notion rested on Shulman's claim that PCK, combined with subject knowledge and curriculum knowledge, formed critical knowledge bases for understanding and improving subject-specific teaching. While subject matter knowledge and PCK are frequently dealt with together in research studies, interest, and contestation in the boundary leads to separate but related entries for them in this encyclopedia (see ► Subject Matter Knowledge Within "Mathematical Knowledge for Teaching" entry). PCK studies in mathematics education indicate attempts at: (a) sharpening theorizations of PCK, (b) measuring PCK, and (c) using notions of PCK to build practical skills within teacher education, or combinations of these elements. This entry summarizes key work across these groups.

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Theorizations of PCK

Key writings in the category of sharpening theorizations of PCK examine both the boundary between PCK and the broader field of subject-related knowledge – sometimes referred to as "Mathematics knowledge for teaching" (MKT), and inwards at subcategories within PCK.

Deborah Ball and the Michigan research group sharpened the distinctions between content knowledge and PCK in their theorization based on the classroom practices of expert teachers: "Subject Matter Knowledge" (SMK) broke down into: common content knowledge (CCK), specialized content knowledge (SCK), and horizon knowledge; and PCK into: knowledge of content and students (KCS), knowledge of content and teaching, and knowledge of curriculum (Ball et al. 2008).

Critiques of work drawing from Shulman's categorizations argue that the "static" conceptualization of MKT with separate components is unhelpful in relation to the interactive and dynamic nature of MKT. Centrally, these critiques argue that MKT is better interpreted as an attribute of pedagogic practices in specific contexts and related to specific mathematical ideas, rather than a generalized attribute of the teacher. Fennema and Franke's (1992) conceptualization of MKT as constituted by knowledge of mathematics, combined with PCK comprised of elements of knowledge of learners' mathematical cognition, pedagogical knowledge, and beliefs views this combination as a taxonomy that can identify the "context-specific knowledge" of a teacher, rather than a more generalized picture of the teacher's MKT. Rowland et al. (2003) similarly emphasize, in their "Knowledge Quartet" formulation consisting of Foundation, Transformation, Connection, and Contingency knowledge (the latter three relating to PCK) that the profile of MKT produced is a categorization of teaching situations, rather than of teachers. Blömeke et al. (2015) framework of teacher competence also explicitly views teacher competences as the outcome of interaction between personal, situational, and social features.

While all of these models were developed from studies of practice, Fennema and Franke and Rowland et al.'s models include a beliefs component, or an affect component in BlÖmeke et al.'s case – which does not feature in Ball et al.'s conceptualization.

Other studies have looked at PCK in alternative formulations (e.g., Silverman and Thompson 2008), with the notion of "connections" within mathematics and with learning (Askew et al. 1997; Ma 1999) seen as critical. Petrou and Goulding (2011) provide an overview of key writings in the MKT field.

Measuring PCK

Ball's research group shifted their attention into measuring MKT to verify assumptions about its relationship to teaching quality and student learning (Ball et al. 2005). The group developed multiple choice items based on specific MKT subcomponents that were administered to teachers, with data collected on their elementary grade classes' learning backgrounds and learning gains across a year. Hill et al.'s (2005) analysis showed content knowledge measures across the common and specialized categories as significantly associated with learning gains. While Ball's group conceptualizes CCK and SCK as part of content knowledge, the descriptions of SCK that are provided – e.g., understanding of representations and explanations – fall within other writers' conceptualizations of PCK.

Baumert et al. (2010), noting the absence of direct attention to teaching in Ball et al.'s measurement-oriented work, developed the COACTIV framework – that distinguished content knowledge from PCK and examined the relationships between content knowledge, PCK, classroom teaching, and student learning gains in Germany. In the COACTIV (Professional Competence of Teachers, Cognitively Activating Instruction, and the Development of Students' Mathematical Literacy) model (focused on secondary mathematics teaching), content knowledge is understood as "a profound mathematical understanding of the mathematics taught at school" (p. 142), and PCK is subdivided into: knowledge of mathematical tasks as instructional tools, knowledge of students' thinking and assessment of understanding, and knowledge of multiple representations and explanations of mathematical problems. With this distinction, separate content knowledge and PCK open response items were developed and administered to nearly 200 teachers in different tracks of the German schooling system. Mathematics test performance data were gathered from over 4000 students in these teachers' classes. Instructional quality was measured through three data sources. The first encompassed selected class, homework, test and examinations tasks, and the degree of alignment between assessment tasks and the Grade 10 curriculum. The second source considered the extent of individual learning support, measured through student rating scales. The third source examined classroom management as degree of agreement between teacher and student perceptions about disciplinary climate.

Baumert et al.'s findings suggested that their theoretical division of content knowledge and PCK was empirically distinguishable, with their PCK variable showing more substantial associations with student achievement and instructional quality than their content knowledge variable.

Using PCK to Support the Development of Pedagogic Practice

The third category of PCK literature links to studies of teacher development using PCK frameworks. This strand often uses longitudinal case study methodologies.

Fennema and Franke, and Rowland's MKT models have associated development-focused studies. Turner and Rowland (2011) provide examples of the Knowledge Quartet's use in England to stimulate development of teaching, and Fennema and Franke, with colleagues, have produced studies on the longevity of the PCK aspects presented within professional development programs.

This category too contains other studies drawing on aspects of PCK. Kinach (2002) focuses on secondary mathematics teachers' development of instructional explanations – a key feature of PCK across different formulations. Learning studies interventions (Lo and Pong 2005) focus on building teachers' awareness of the relationship between particular objects of learning and students' work with these objects – a feature of the KCS terrain.

Emerging Directions

Emerging work reflects, in some ways, an increasingly polarized world. One line of PCK research is focused on pedagogical technological knowledge: teachers' awareness and competence with integrating technology into their mathematics teaching in ways that support learning (Clark-Wilson et al. 2014). Another line of research questions the assumption of basic coherence and connection in MKT that underlies much of the PCK writing (Silverman and Thompson 2008). Frameworks developed from qualitative case studies of classroom teaching detail inferences relating to PCK (and SMK) in contexts of pedagogic fragmentation and disconnections, where, as Askew (2018) notes, assumptions of "a baseline of mathematical coherence in lessons . . . is not yet in place."

PCK as a field therefore continues to thrive, in spite of ongoing differences in nomenclature, underlying views about specific subelements, and the nature of their interaction.

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Cross-References

► Subject Matter Knowledge Within "Mathematical Knowledge for Teaching"

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Policy Debates in Mathematics Education

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Definition

Policy in mathematics education concerns the nature and shape of the mathematics curriculum, that is, the course of study in mathematics of a school or college. This is the teaching sequence for the subject as planned and experienced by the learner. Four aspects can be distinguished, and these are the focuses of policy debates:

1. The aims, goals, and overall philosophy of the curriculum
2. The planned mathematical content and its sequencing, as in a syllabus
3. The pedagogy employed by teachers
4. The assessment system

History

The New Math debate of the late 1950s to the mid-1960s was primarily about the content of the mathematics curriculum. At that time traditional school mathematics did not incorporate any modern topics. The content consisted primarily of arithmetic at elementary school, plus traditional algebra, Euclidean geometry, and trigonometry at high school. The New Math curriculum broadened the elementary curriculum to include other aspects of mathematics, and high school mathematics incorporated modern algebra (including sets, functions, matrices, vectors), statistics and probability, computer mathematics (including base arithmetic), and modern geometry (transformation geometry, topological graph theory). The launch of Sputnik, the first earth orbiting satellite, by the Soviet Union in 1957, during the Cold War led to fears that the USA and UK were being overtaken in technology and in mathematics and science education by the Soviets. Government funding became available, especially in the USA, to extend projects modernizing the mathematics curriculum in a bid to broaden and improve students' knowledge of mathematics, such as the Madison Project in 1957 and The School Mathematics Study Group in 1958 in the USA. In the UK independent curriculum projects emerged, including the School Mathematics Project (SMP) in 1961 and Nuffield Primary Mathematics in 1964. These projects did not cause much controversy at the national policy levels although there was a concern by parents that they did not understand the New Math their children were learning. The relatively muted debates concerned the changing content of the mathematics curriculum rather than its pedagogy or assessment.

In the mid to late 1960s onwards a new debate emerged about discovery learning. In the UK the Schools Council Curriculum Report No. 1 (Biggs 1965) on the teaching and learning of mathematics in primary school proposed practical approaches and "discovery learning" as the most effective ways of teaching mathematics. Sixty-five percent of all primary teachers in the UK read Biggs (1965), and it had a significant impact. Discovery learning was a central part of the 1957 Madison Project developed by Robert B. Davis. This and similar developments led to a major policy debate on discovery learning. Is discovery learning the most effective way to learn mathematics? Proponents of discovery contrasted it with rote learning. Self-evidently rote learning cannot be the best way to learn all but the simplest mathematical facts and skills since it means simply "learning by heart." However, educational psychologist Ausubel (1968) argued successfully that discovery and rote learning are not part of a continuum but on two orthogonal axes defined by pairs of opposites: meaningful versus rote learning and reception versus discovery learning. Meaningful learning is linked to existing knowledge; it is relational and conceptual. Rote learning is arbitrary, verbatim, and disconnected – unrelated to other existing knowledge of the learner. Knowledge learned by reception comes already formulated and is acquired through communication, such as in expository teaching or reading. Ausubel distinguishes this from discovered knowledge that has to be formulated by the learner herself.

The promotion of discovery learning led to heated debate on both sides of the Atlantic. Shulman and Keislar (1966) offered a review, but to this day the evidence remains equivocal. This debate was primarily about pedagogy – how best to teach mathematics. But underneath this debate one can discern battle lines being drawn between a child-centered, progressive ideology of education with roots going back to Rousseau, Montessori, Dewey, and a traditionalist teacher- and knowledge-centered ideology of education favored by some mathematicians and university academics.

The mid-1970s saw the birth of the back-to-basics movement promoting basic arithmetical skills as the central goal of the teaching and learning of mathematics for the majority. This was a reaction to the progressivism of the previous decade, most clearly defined in the aims of the Industrial Trainers group mentioned below, and became an important plank of the traditionalist position on the mathematics curriculum.

The early 1980s led to a further entrenchment in the progressive/traditional controversy. In the USA the influential National Council of Teachers of Mathematics (NCTM) recommended that "Problem solving must be the focus of school mathematics in the 1980s" (1980, pp. 2–4). In the UK the Cockcroft Inquiry (1982) recommended problem solving and investigational work be included in mathematics for all students. Thus the debate remained at the level of pedagogy but shifted to problem solving.

The progressivist versus traditionalist debate was born anew in the late 1980s (UK) and the 1990s (USA) but now encompassed the whole mathematics curriculum on a national basis.

Analytical Framework

The British government developed and installed the first legally binding National Curriculum in 1988 for all students age 5–16 years in all state schools (excluding Scotland). The debate over the mathematics part of National Curriculum in became a heated contest between different social interest groups. Ernest (1991) analyzed this as a contest between five different groups with different broad ranging ideologies of education, the aims, and orientation of which are summarized in Table 1 (In the full treatment there are 14 different ideological components for each of these 5 groups).

These different social groups were engaged in a struggle for control over the National Curriculum in mathematics, since the late 1980s (Brown 1996). In brief, the outcome of this contest was that the first three more reactionary groups managed to win a place for their aims in the curriculum. The fourth group (progressive educators) reconciled themselves with the inclusion of a personal knowledge-application dimension, namely, the processes of "Using and Applying mathematics," constituting one of the National Curriculum assessment targets. However instead of representing progressive self-realization through creativity aims through mathematics, this component embodies utilitarian aims: the practical skills of being able to apply mathematics to solve work-related problems with mathematics. Despite this concession over the nature of the process element included in the curriculum, the scope of the element has been reduced over successive revisions that have occurred in the subsequent 20 years and has largely been eliminated. The fifth group, the public educators, found their aims played no part in the National Curriculum. The outcome of the process is a largely utilitarian mathematics curriculum developing general or specialist mathematics skills and capabilities, which are either decontextualized – equipping the learner with useful tools – or which are applied to practical problems. The contest between the interest groups was an ideological one, concerning not only all four aspects of curriculum but also about deeper epistemological theories on the nature of mathematics and the nature of learning.

During the period following the introduction of the National Curriculum in mathematics, pressure from various groups continued to be exerted to shift the emphasis of the curriculum. Mathematicians who can often be characterized as belonging to the Old Humanist grouping published a report entitled *Tackling the Mathematics Problem* (London Mathematical Society 1995), commissioned by professional mathematical organizations. This criticized the inclusion of "time-consuming activities (investigations, problem solving, data surveys, etc.)" at the expense of "core" technique and technical fluency. Furthermore, it claimed many of these activities are poorly focused and can obscure the underlying mathematics. This criticism parallels that heard in the "math wars" debate in the USA.

Policy Debates in Mathematics Education, Table 1 Five interest groups and their aims for mathematics teaching. (Based on Ernest 1991)

Interest group	Social location	Orientation	Mathematical aims
1. Industrial trainers	Radical New Right conservative politicians and petty bourgeois	Authoritarian, basic skills centered	Acquiring basic mathematical skills and numeracy and social training in obedience
2. Technological pragmatists	Meritocratic industry-centered industrialists, managers, etc., New Labor	Industry and work centered	Learning basic skills and learning to solve practical problems with mathematics and information technology
3. Old Humanist mathematicians	Conservative mathematicians preserving rigor of proof and purity of mathematics	Pure mathematics centered	Understanding and capability in advanced mathematics, with some appreciation of mathematics
4. Progressive educators	Professionals, liberal educators, welfare state supporters	Child-centered progressivist	Gaining confidence, creativity, and self-expression through maths
5. Public educators	Democratic socialists and radical reformers concerned with social justice and inequality	Empowerment and social justice concerns	Empowerment of learners as critical and mathematically literate citizens in society

"Math Wars"

In the USA the National Council of Teachers of Mathematics (NCTM) published its so-called Standards document in 1989 recommending a "Reform"-based (progressive) mathematics curriculum for the whole country. This emphasized problem solving and constructivist learning theory. The latter is not just discovery learning under a new name because constructivists acknowledge that learners need to be presented with representations of existing mathematical knowledge to reconstruct them for themselves. This initiated the savage debate in the USA called the Math Wars (Klein 2007).

The Standards influenced a generation of new mathematics textbooks in the 1990s, often funded by the National Science Foundation. Although widely praised by mathematics educators, particularly in California, concerned parents formed grassroots organizations to object and to pressure schools to use other textbooks. Reform texts were criticized for diminished content and lack of attention to basic skills and an emphasis on progressive pedagogy based on constructivist learning theory. Critics in the debate derided mathematics programs as "dumbed-down" and described the genre as "fuzzy math."

In 1997 Senator Robert Byrd joined the debate by making searing criticisms of the mathematics education reform movement from the Senate floor focusing on the inclusion of political and social justice dimensions in one mathematics textbook. In the spreading and increasingly polarized debate, the issues spread from traditional versus progressive content and pedagogy to left versus right political orientations and traditional objectivist versus constructivist (relativist) epistemology and philosophy of mathematics. This way the debate took on aspects of the parallel "science wars" also taking place, primarily in the USA. This is the heated debate between scientific realists, who argued that objective scientific knowledge is real and true, and sociologists of science. The latter questioned scientific objectivity and argued that all knowledge is socially constructed. This is an insoluble epistemological dispute that has persisted at least since the time of Socrates in philosophical debates between skeptics and dogmatists. Nevertheless, it fanned the flames of the Math Wars debate.

In 1999 the US Department of Education released a report designating 10 mathematics programs as "exemplary" or "promising." Several of the programs had been singled out for criticism by mathematicians and parents. Almost immediately an open letter to Secretary of Education Richard Riley was published calling on him to withdraw these recommendations. Over 200 university mathematicians signed their names to this letter and included seven Nobel laureates and winners of the Fields Medal. This letter was repeatedly used by traditionalists in the debate to criticize Reform mathematics, and in 2004 NCTM President Johnny Lott posted a strongly worded denunciation of the letter on the NCTM website.

In 2006, President Bush was stirred into action by the heated controversy and created the National Mathematics Advisory Panel to examine and summarize the scientific evidence related to the teaching and learning of mathematics. In their 2008 report, they concluded that recommendations that instruction should be entirely "student centered" or "teacher directed" are not supported by research. High-quality research, they claimed, does not support the exclusive use of either approach. The Panel called for an end to the Math Wars, although its recommendations were still the subject of criticism, especially from within the mathematics education community for its comparison of extreme forms of teaching and for the criteria used to determine "high-quality" research.

Defusing the Debates

Policy debates have raged over the mathematics curriculum throughout the past 50 years. They have been strongest in the USA and UK but have occurred elsewhere in the world as well. In Norway, for example, there is a much more muted but still heated debate as to whether mathematics or the child should be the central focus of the curriculum. Proponents of a child-centered curriculum promote general pedagogy in teacher education as opposed to a specifically mathematics pedagogy with its associated emphasis on teachers' pedagogical content knowledge in mathematics.

The spread of policy debates has also become much wider following the impact of international assessment projects such as TIMSS. Politicians sometimes blame what is perceived to be poor national performance levels in mathematics on one or other aspect of the curriculum. Unfortunately policy debates too often become politicized and drift away from the central issues of determining the best mathematics curriculum for students. In becoming polarized, the debates become controversies that propel policy swings from one extreme to the other, like a pendulum. Ernest (1989) noted this pattern, but regrettably the pendulum-like swings from one extreme position to the opposite continue unabated to this day.

The fruitlessness of swings from traditional to progressive pedagogy in mathematics is illustrated in an exemplary piece of research by Askew et al. (1997). This project studied the belief sets and teaching practices of primary school teachers and their correlation with students' numeracy scores over a period of 6 months. Three belief sets and approaches to teaching numeracy were identified in the teachers:

1. Connectionist beliefs: valuing students' methods and teaching with emphasis on establishing connections in mathematics (mathematics and learner centered)
2. Transmission beliefs: primacy of teaching and view of maths as collection of separate routines and procedures (traditionalist)
3. Discovery beliefs: primacy of learning and view of mathematics as being discovered by students (progressivist)

The classes of teachers with a connectionist orientation made the greatest gains, so teaching for connectedness were measurably the most effective methods. This included attending to and valuing students' methods as well as teaching with an emphasis on establishing connections in mathematics. Traditional transmission beliefs and practices were not shown to be as effective. Likewise, discovery beliefs and practices were equally ineffective, refuting the progressivist claim that the teaching and learning of mathematics by discovery is the most effective approach. Of course Askew et al. (1997) only report a small-scale, in-depth study of about 20 teachers and must be viewed with caution and needs replication. Nevertheless its results illustrate the futility of policy debates becoming overly ideological and losing contact with empirical measures of effectiveness from properly conducted research.

Cross-References

▶ Authority and Mathematics Education
▶ Critical Mathematics Education
▶ Dialogic Teaching and Learning in Mathematics Education
▶ History of Mathematics Teaching and Learning
▶ Inquiry-Based Mathematics Education
▶ Mathematical Literacy
▶ Political Perspectives in Mathematics Education
▶ Realistic Mathematics Education
▶ Recontextualization in Mathematics Education
▶ Teacher-Centered Teaching in Mathematics Education

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Political Perspectives in Mathematics Education

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Keywords

Power · Politics · Modernity · Neutrality of mathematics · Social rationalities · In(ex)clusion · Mathematics for all · Credit system · Subjectivity

Definition

A political perspective in mathematics education is a way of looking at how mathematics, education, and society relate to power. It stands on the critical recognition that mathematics is not only important in society due to its exceptional, intrinsic characteristics as the purest and most powerful form of abstract thinking but also and foremost, because of its functionality in the constitution of the dominant cultural project of Modernity. Thus, it assumes that the teaching and learning of mathematics are not neutral practices but that they insert people – be it children, youth, teachers, and adults – in socially valued mathematical rationalities and forms of knowing. Such insertion is part of larger processes of selection of people that schooling operates in society. It results in differential positioning of inclusion or exclusion of learners in relation to access to socially privileged resources such as further education, labor market, and cultural goods.

History

The political perspectives of mathematics education became a concern for teachers and researchers in the 1980s. While the change from the nineteenth to the twentieth centuries was a time of inclusion of mathematics in growing, massive, national education systems around the world, the change from the twentieth to the twenty-first centuries has been a time for focusing on the justifications for the privileged role of mathematics in educational systems at all levels. The apparent failure of the New Math movement in different industrialized countries allowed to raise concerns about the need for mathematics teaching and learning that could reach as many students as possible and not only a selected few (Damerow et al. 1984). Questions of how mathematics education could be studied from perspectives that allowed moving beyond the boundaries of the mathematical contents in the school curriculum started to be raised. In mathematics education, the first book published in English as part of an international collection, containing the word politics in the title, was *The politics of mathematics education* by Stieg Mellin-Olsen (1987). However, *The mastery of reason: Cognitive development and the production of rationality* by Valerie Walkerdine (1988) is a seminal work in critical psychology discussing how school mathematics education subjectifies children through inscribing in them and in society, in general, specific notions of the rational child and of abstract thinking.

The political concern and involvement of many mathematics educators in their teaching and research practice was also an initial entry that allowed sensitivity and awareness for searching how mathematics education could be "political" (Lerman 2000). Such political awareness on issues such as how mathematics has played a role as gatekeeper to entry in further education, for example, has been important. However, a political "awareness" does not constitute the center of a political approach since there is a distinction between being sympathetic to how mathematics education relates to political processes of different type and making power in mathematics education the focus of one's research. In other words, not all people who express a political sympathy actually study the political in mathematics education (Gutierrez 2013; Valero 2004).

With this central distinction in mind, it is possible to differentiate a variety of political perspectives, some that could be called *weak* in the sense that they make a connection between mathematics education and power but do not concentrate on the study of it as a constituent of mathematics education but rather as a result or a simply associated factor. *Strong* political approaches in mathematics education are a variety of perspectives that do have a central interest in understanding mathematics education as political practices.

Weak Political Perspectives

A general characteristic of weak political perspectives in mathematics education is the adherence to some of the positive features attributed to mathematics and mathematics education, particularly those that have to do with people's empowerment and social and economic progress. More often than not, these views assume some kind of intrinsic goodness of mathematics and mathematics education that is transferred to teachers and learners alike through good and appropriate education practices. In the decade of the 1980s and fully in the 1990s, the broadening of views on what constitutes mathematics education allowed for formulations of the aims of school mathematics in relation to the response to social challenges of changing societies and, in particular, in response to the consolidation of democracy. It was possible to enunciate the idea that, as part of a global policy of "Education for all" by UNESCO, mathematics education had to contribute to the competence of citizens, but also to open access for all students. In many countries, both at national policy level and at the level of researchers and teachers, there was a growing concern for mathematics for all and mathematics for equity and inclusion. Since the 2000s, the growing emphasis given to mathematical achievement as an indicator of economic growth among international, competitive economies has reinforced the idea of the power of mathematical competence to improve citizens' life chances and national economic progress. The study of how different groups – women, language, ethnic or religious minorities, and particular racial groups – of students systematically underachieve and how to remediate that situation grew extensively. While this type of studies emerged mainly in English-speaking countries, there is a growing tendency to see mathematical underachievement as a national and international concern and therefore many studies are being carried out in different countries to generate inclusion of different types of students in and through mathematics education. The impact of international comparative assessments such as TIMMS and PISA are connected to this trend.

Part of the weak political approaches also includes studies of how mathematics education practices are shaped by educational policies. South Africa, given the transition from apartheid to democracy at the beginning of the 1990s, has been a particularly interesting national case where deep changes of policy had been studied to see how and why mathematics education in primary and secondary school is transforming to contribute – or not – to the construction of a new society. Since the 2000s the concern with inclusion as a way of facing systematic low performance in mathematics has promoted government-promoted large-scale pedagogical interventions as well as small-scale pedagogical innovations. Many of these studies have a weak political approach in the sense that they are justified and operate on some political assumptions on mathematics education and its role in society, but intend to study appropriate pedagogies and not how pedagogies in themselves effect the exclusion that the programs intend to remediate.

Strong Political Perspectives

Strong political perspectives in mathematics education problematize the assumed neutrality of mathematical knowledge and provide new interpretations of mathematics education as practices of power. Ethnomathematics can be read as a political perspective in mathematics education in its challenge to the supremacy of Eurocentric understandings of mathematics and mathematical practices. The strong political perspectives of ethnomathematics are presented in studies that not only argue for how the mathematical practices of different cultural groups – not only indigenous or ethnic groups but also professional groups – are of epistemological importance and value but also how some of those cultural practices are inserted in the calculations of power so that they can construct a regime of truth around themselves and thus gain a privileged positioning in front of other practices (Knijnik 2012).

Critical mathematics education as a wide and varied political approach takes the study of power in relation to how mathematics is a formatting power in society through its immersion in the creation of scientific and technological structures that operate in society (Christensen et al. 2008). It also studies the processes of exclusion and differentiation of students when mathematics education practices reproduce the position of class and disadvantage of students (Frankenstein 1995), and when such reproduction is part of the way (school) mathematics is given meaning in public discourses and popular culture (Appelbaum 1995). It also offers possibilities for rethinking practices when democracy is thought as a central element of mathematics education (Skovsmose and Valero 2008).

The study of the political in relation to the alignment of mathematics education practices with Capitalism is also a recent and strong political reading of mathematics education that offers a critical perspective on the material, economic significance of having success in mathematics education. Both educational practices (Baldino and Cabral 2006) and research practices (Lundin 2012; Pais 2012) lock students in a credit system where success in mathematics represents value.

In the USA, and as a reaction to endemic operation of race as a strong element in the classification of people's access to cultural and economic resources, the recontextualization of critical race theories into mathematics education has provided new understandings of mathematics education as a particular instance of a White-dominant cultural space that operates exclusion from educational success for African American learners (Martin 2011), as well as for Latino(a)s (Gutiérrez 2012).

The recontextualization of poststructural theories in mathematics education has also led to the study of power in relation to the historical construction of Modern subjectivities. The effects of power in the bodies and minds of students and teachers (Walshaw 2010), as well as in the public and media discourses on mathematics (Moreau et al. 2010), are studied in an attempt to provide insights into how the mathematical rationality that is at the core of different technologies in society shapes the meeting between individuals and their culture. Even though most research concentrates on the issue of identity construction and subjectivity, some studies attempting cultural histories of mathematics as part of Modern, massive educational systems are also broadening this type of political perspective (Popkewitz 2004; Valero et al. 2012).

Recent Overviews

As the mathematical qualification of the population is seen in the 2010s as a matter of economic development, weak and strong political perspectives in mathematics education become solidly rooted in society. A growing number of studies continue to explore the contexts in which mathematics education is understood in relation to power in contemporary societies. Recent overviews (Jurdak et al. 2016) offer a landscape of the issues and preoccupations addressed when adopting political perspectives in mathematics education.

Cross-References

- ▶ Equity and Access in Mathematics Education
- ▶ Mathematization as Social Process
- ▶ Policy Debates in Mathematics Education
- ▶ Poststructuralist and Psychoanalytic Approaches in Mathematics Education
- ▶ Socioeconomic Class and Socioeconomic Status in Mathematics Education
- ▶ Sociological Approaches in Mathematics Education

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Poststructuralist and Psychoanalytic Approaches in Mathematics Education

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Keywords
Poststructuralism · Psychoanalysis

Definition

Approaches that draw on developments within wider scholarly work that conceives of modernist thought as limiting.

Characteristics

Poststructuralist and psychoanalytic approaches capture the shifts in scholarly thought that gained currency in Western cultures during the past 50 years. Conveying a critical and self-reflective attitude, both raise questions about the appropriateness of modernist thinking for understanding the contemporary social and cultural world. Since the publication of Lyotard's *The Postmodern*

Condition (translated into English in 1984), poststructuralist and psychoanalytic thinking have provided an expression within the social sciences and humanities and, more recently, within mathematics education, for a loss of faith in the "grand narratives" of Western history and, in particular, enlightened modernity. A diverse set of initiatives in social and philosophical thought, originating from the work of Michel Foucault (e.g., 1970), Jacques Derrida (e.g., 1976), Julia Kristeva (e.g., 1984), and Jacques Lacan (1977), among others, helped crystallize poststructuralist and psychoanalytic ideas among researchers and scholars within mathematics education about how things might be thought and done differently.

Poststructuralist and psychoanalytic approaches provide alternatives to the traditions of psychological and sociological thought that have grounded understandings about knowledge, representation, and subjectivity within mathematics education. These traditions understand reality as characterized by an objective structure, accessed through reason. More specifically, the traditions are based on the understanding that reason can provide an authoritative, objective, true, and universal foundation of knowledge. They also assume the transparency of language. Epistemological assumptions like these, about the relationship between the knower and the known, are accompanied by beliefs about the kind of being the human is. Typically, the related ontologies are dualist in nature. They include such dichotomies as rational/irrational, objective/subjective, mind/body, cognition/affect, and universal/particular. Taken together, these characteristically modernist beliefs about ontology and epistemology have informed theories of human interaction, teaching, learning, and development within mathematics education.

Developments within psychology and sociology that began to question these understandings paved the way for a different perspective. Sociology has helped seed poststructuralist work that aims to draw attention to the ways in which power works within mathematics education, at any level, and within any relationship, to constitute identities and to shape proficiencies. Psychology has informed a psychoanalytical turn, designed to unsettle fundamental assumptions concerning identity formations. Postmodernists and psychoanalysts share some fundamental assumptions about the nature of the reality being studied, assumptions about what constitutes knowledge of that reality, and assumptions about what are appropriate ways of building knowledge of that reality.

Researchers in mathematics education who draw on this body of work have an underlying interest in understanding, explaining, and analyzing the practices and processes within mathematics education. Their analyses chart teaching and learning, and the way in which identities and proficiencies evolve; tracking reflections; investigating everyday classroom planning, activities, and tools; analyzing discussions with principals, mathematics teachers, students, and educators; mapping out the effects of policy, and so forth. In the process of deconstructing taken-for-granted understandings, they reveal how identities are constructed within discourses, they demonstrate how everyday decisions are shaped by dispositions formed through prior events, and they provide insights about the way in which language produces meanings and how it positions people in relations of power. The assumptions upon which these analyses are based enable an exploration of the lived contradictions of mathematics processes and structures.

These analyses are developed around a number of key organizing principles: language is fragile and problematic and constitutes rather than reflects an already given reality. Meaning is not absolute in relation to a referent, as had been proposed by structuralism. The notion of knowing as an outcome of human consciousness and interpretation, as described by phenomenology, is also rejected. Moreover, knowing is not an outcome of different interpretations, as claimed by hermeneutics. Instead, for poststructuralist and psychoanalytic scholars, reality is in a constant process of construction. What is warranted at one moment of time may be unwarranted at another time. The claim is that because the construction process is ongoing, no one has access to an independent reality. There is no "view from nowhere," no conceptual space not already implicated in that which it seeks to interpret. There is no stable unchanging world and no realm of objective truths to which anyone has access. The notion of a disembodied autonomous subject with agency to choose what kind of individual he or she might become also comes under scrutiny. The counter-notion proposed is a "decentered" self – a self that is an effect of discourse which is open to redefinition and which is constantly in process.

Poststructuralist Approaches

Foucault's work is considered by many to represent a paradigmatic example of poststructuralist thought. His work raises critical concerns about how certain practices, and not others, become intelligible and accepted, and how identities are constructed. Foucauldian analyses centered within mathematics educational sites explore lived experience, not in the sense of capturing reality and proclaiming causes but of understanding the complex and changing processes by which subjectivities and knowledge production are shaped. In that sense, the focus shifts from examining the nature of identity and knowledge to a focus on how identity and knowledge are discursively produced. In these analyses, "discourse" is a key concept. Discourses sketch out, for teachers, students, and others, ways of being in the classroom and within other institutions of mathematics education. They do that by systematically constituting specific versions of the social and natural worlds for them, all the while obscuring other possibilities from their vision. Discursivity is not simply a way of organizing what people say and do; it is also a way of organizing actual people and their systems. It follows that "truths" about mathematics education emerge through the operation of discursive systems.

Discursive approaches within mathematics education draw attention to the impact of regulatory practices and discursive technologies on the constructions of teachers, students, and others. It reveals the contradictory realities of teachers, students, policy makers, and so forth and the complexity and complicity of their work. Such work emphasizes that teachers and students are the production of the practices through which they become subjected (e.g., Hardy 2009; Lerman 2009).

Power in these approaches envelopes everyone. What the analyses reveal is that, in addition to operating at the macro-level of the school, power seeps through lower levels of practice such as within teacher/student relations and school/teacher relations (see Walshaw 2010). Even in a classroom environment that provides equitable and inclusive pedagogical arrangements, poststructural approaches have shown that power is ever present through the classroom social structure, systematically creating ways of being and thinking in relation to class, gender, and ethnicity and a range of other social categories (see Walshaw 2001; Mendick 2006; Knijnik 2012).

In illuminating the impact of regulatory practices and technologies on identity and knowledge production, fine-grained readings of classroom interaction have revealed the regulatory power of teachers' discourse in providing students with differential access to mathematics (de Freitas 2010). Such readings shed light on how the discursive practices of teachers contribute to the kind of mathematical thinking and the kind of mathematical identities that are possible within the classroom.

Psychoanalytic Approaches

Psychoanalytic analyses in mathematics education explore the question of identity. Lacan's (e.g., 1977) and Žižek's (e.g., 1998) explanations of how identities are constructed through an understanding of how others see that person have been influential in revealing that teachers, students, and others are not masters of their own thoughts, speech, or actions. Žižek's psychoanalytic position is that the self is not a center of coherent experience: "there are no *identities* as such. There are just *identifications* with particular ways of making sense of the world that shape that person's sense of his self and his actions" (Brown and McNamara 2011, p. 26). A person's identifications are not reducible to the identities that the person constructs of himself. Rather, the self is performed within the ambivalent yet simultaneous relationship of subjection/agency.

Psychoanalytic observations of identity formation are likely to reveal how identities develop through discourses and networks of power that shift continually in a very unstable fashion, changing as alliances are formed and reformed. When identities are formed in a very mobile space, what emerge are fragmented selves, layers of self-understandings, and multiple positionings within given contexts and time (see Hanley 2010). This psychoanalytic idea is fundamental to understanding that teachers and students (among others) negotiate their way through layered meanings and contesting perceptions of what a "good" teacher or student looks like. To complete a negotiation, there is a level at which the teacher or student invests, or otherwise, in a discursive position made available (see Bibby 2009).

A teacher's, for example, investments within one discourse rather than another is explained through the notion of affect and, more especially, through the notions of obligation and reciprocity. Affect, in the psychoanalytic analysis, is not a derivative aspect but a constitutive quality of classroom life (see Walshaw and Brown 2012). It is not an interior experience, but rather, it operates through processes that are historical in a way that is not entirely rational nor observable. Researchers in mathematics education who draw on psychoanalytic theory maintain that determinations exist outside of our consciousness and, in the pedagogical relation, for example, influence the way teachers develop a sense of self as teacher and influence their interactions in the classroom. The identities teachers have of themselves are, in a very real sense, "comprised," made in and through the activities, desires, interests, and investments of others. Understandings like these invite unknowingness, fluidity, and becoming, which, in turn, have the effect of producing different knowledge.

Emancipatory Possibilities

Although both poststructuralist and psychoanalytic theorists question the modernist concept of enlightenment, in reconceptualizing emancipation away from individualist sensibilities, they highlight possibilities for where and in what ways mathematics educational practices might be changed (see Radford 2012). In addition to uncovering terrains of struggle, poststructuralist and psychoanalytic analyses foster democratic provision, enabling a vision of critical-ethical teaching where different material and political conditions might prevail. What is clarified in these approaches is that discourses are not entirely closed systems but are vehicles for reflecting on where mathematics education is today, how it has come to be this way, and the consequences of conventional thought and actions. Importantly, such analyses are a political resource for transforming the processes and structures that currently deny teachers, students, policy makers, and others the achievement of their ethical goals within mathematics education.

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▶ Psychological Approaches in Mathematics Education
▶ Sociological Approaches in Mathematics Education

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Preparation and Professional Development of University Mathematics Teachers

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Keywords

University mathematics teaching development · Mathematics pedagogy · Mathematics support · Teacher learning · Developmental research initiatives

Introduction

University mathematics teachers are the people who teach mathematics in a university to mathematics undergraduates, students in science or engineering, business or economics, and in foundation programs where these exist. They may be research mathematicians or mathematics educators, or mathematics teachers who do not engage in research.

In the following sections, the needs for and means of developing professionally for both practicing and prospective teachers are discussed: some as part of university professional development programs, others as part of research studies or teaching initiatives providing new learning and development opportunities.

What Is Included in Preparation and Professional Development?

A critical review of teaching at university level suggested that "many academics have had little or no formal teacher education to prepare them for the teaching role" (Kane et al. 2002). Oleson and Hora (2014) recognize that forms of teaching in Higher Education are recycled by successive generations of teachers despite changes in curriculum or student population. Where mathematics education is concerned, Nardi et al. (2005) pointed out: "Teachers of university mathematics courses, on the whole, have not been trained in pedagogy and do not often consider pedagogical issues beyond the determination of syllabus; few have been provided with incentives or encouragement to seek out the findings of research in mathematics education" (p. 284). A recent survey of research (Winsløw et al. (2018) has recognized a scarcity of research into university teachers' pedagogical knowledge and its development through formalized education), stating that "organised, deliberate development of UME teachers, based on RUME, is still rare" (p. 70).

The need for development is clearly urgent when we consider the changing nature of the student population. More students are going to university than ever before, from a wide range of backgrounds, many of whom are likely not to have had prior mathematics teaching that enables them to tackle the more abstract and formal modes of thinking required in university mathematics (Nardi 2008; Hawkes and Savage 2000). Teachers have to acknowledge that large audience lectures in early university courses (see ▶ "Teaching Practices at University Level"), and the amount of material which is presented in a typical university course (see ▶ "University Mathematics Education") create problems for students. In addition, the so-called "service teaching," for students of science, engineering, economics, and so on, needs alternative teaching practices related to the needs and interests of these students (see ▶ "Service-Courses in University Mathematics Education").

University Provision for Both New and Experienced Teachers of Mathematics

Despite these evident limitations, we see also an increasing awareness of the importance of preparation and PD for all those involved in teaching mathematics at university level. A very recent example was seen at the INDRUM II (International Network for Didactic Research in University Mathematics) conference, held in April 2018, where a plenary panel focused on the *Education and Professional Development of University Mathematics Teachers*. Contributors from Germany, Norway, the UK, and the USA focused on professional development provision in their national settings (Winslow et al. 2019). While only four countries were represented, the characteristics emerging and issues identified seemed relevant to a wider constituency. For example, the US participant claimed that, "In the US, there is no universal professional development for university mathematics professors related to the professional activity of teaching." The same was acknowledged as true in the other countries. Typically, universities organize their own PD programs for their own staff. In Germany, one university program consisted of three general modules (not specific to mathematics) of 70 hours' workload (Basic, Expansion, and Advanced) focusing on elements of didactics, pedagogy, self-image, and student-support for postdocs and "young" professors. The situation in both Norway and the UK is similar, except that "training" is often for new university teachers (not postdocs). The Norwegian participant suggested that his university's PD program is "too general – [there is] a need for more *subject specific* content."

In Norway, this is being addressed through a new initiative for the preparation of mathematics teachers, organized by the Norwegian National Centre for Teaching Excellence (MatRIC). The elements are: *Topics*: Innovative approaches to teaching, learning and assessing mathematics, and relevant research results; a *Course project*: in which each participant chooses an area of her/his teaching/supervision/presentation activity that she/he would like to develop; a *Professional portfolio*: a structured and organized collection of a range of documentary evidence of professional experiences. Based on this, and satisfactory attendance, a certificate of participation is provided.

In the UK, the government has instituted a Teaching Excellence Framework (TEF) (https://www.officeforstudents.org.uk/advice-and-guidance/teaching/what-is-the-tef/) which measures teaching excellence across a university in three key areas: *Teaching Quality*, *Learning Environment* and *Student Outcomes*. These are general areas not related to a specific subject. Universities may opt in to being evaluated on the framework; one of three levels Gold, Silver, and Bronze is subsequently awarded.

In the USA, there are three national, research-based professional development opportunities:

- Project NEXT – a professional development program organized by the Mathematical Association of America (MAA) for new or recent PhDs in the mathematical sciences (https://www.maa.org/programs-and-communities/professional-development/project-next)
- Inquiry-Based Learning – centralized through the Academy of Inquiry-Based Learning and funded by the US National Science Foundation (http://www.inquirybasedlearning.org/).
- Project TIMES – *T*eaching *I*nquiry-Oriented *M*athematics: *E*stablishing *S*upports (TIMES)

is funded by the US National Science Foundation (http://times.math.vt.edu).

From these examples, it can be seen that much of the existing PD is general and provided by individual universities, in some cases in relation to a national framework. In addition, there are specific programs, developed through national agencies and focusing specifically on aspects of mathematics in higher education. These programs are provided for new teachers or for graduate students/postdocs who wish to gain teaching expertise.

In addition to such programs, we see below further contributions to teacher learning and teaching development.

Learning from Research and Scholarship in Mathematics Education

A number of books are emerging addressing the needs of teachers of mathematics at the university level. The ICMI study into *The Teaching and Learning of Mathematics at University Level* points to a range of perceptions about established mathematics teaching and associated teacher beliefs that impede student learning (Holton 2000). In the study volume, Alsina (2001) suggests "a new paradigm of teaching mathematics at university level" to address context, historical backgrounds, modelling processes, innovative technological tools, pedagogical strategies in mathematics education at the university level (pp. 7–9). These aspirations raise questions about how teachers' beliefs become challenged and practices develop to encompass new possibilities for practice. Other chapters take up these issues.

A book specially written to bring research findings in mathematics education to the attention of university teachers of mathematics addresses specific topics in undergraduate mathematics (e.g., limits and convergence), relating them to research findings on student understanding in advanced mathematics (Alcock and Simpson 2009). An expectation is that mathematicians become more aware of how mathematical topics can be made more accessible to students.

A book written to address a wide range of issues for university teachers of mathematics set out to chart *Transitions in Undergraduate Mathematics Education* (Grove et al. 2015). The chapters include discussions on problem solving and modelling, group work, lecturing, neurodiversity, transition to abstraction in mathematics, gender, and employability. Here teachers of mathematics gain access to range of pedagogies they can develop to improve the student experience.

Teachers' Learning through Engagement in Research and Development Projects

Research into practices in teaching and learning mathematics at university level can have a developmental outcome. Where teachers engage as respondents or participants, they correspondingly learn and develop their teaching overtly or implicitly. A short review of research into university teaching practices pointed to the value for practitioners of research into their practice promoting deeper reflections and potential teaching development (Jaworski et al. 2017). Examples of more overtly developmental studies can be seen in three cases in which researchers and teachers jointly sought to learn about or to develop practice. A project in Denmark studied relationships between research and teaching in mathematics and in geography in which participating teachers gained insights for their own teaching (Madsen and Winslow 2008). The SYMBoL project (Second Year Mathematics Beyond Lectures) in the UK set out to provide resources for courses in which students had been performing at a low level (Duah et al. 2014; Duah 2017). Former students of the courses worked with teachers to design resources to improve conceptual understanding of future students. Through collaboration, students and teachers together learned about the needs of teaching and the design of resources, and teachers gained insights to student perspectives. The third study, a partnership between a mathematician and two mathematics educators to study the teaching of linear algebra, resulted in developing awareness and practice of the teacher whose teaching was observed (Treffert-Thomas 2015).

Innovation in Teaching

Research is starting to be seen into innovative practices in teaching mathematics. A particular kind of collaborative research project explores the integration of new approaches into teaching, or some overt kind of intervention. For example, the ESUM project (Engineering Students Understanding Mathematics) in the UK was designed to improve teaching of mathematics to first year engineering students through a fourfold intervention: use of inquiry-based questions, small group problem solving, a computer-based learning environment, and an assessed group project (Jaworski et al. 2012). The project revealed issues relating to innovation and insights into students' perceptions of learning and teaching, both of which influenced future practice.

A project in France studied university teachers' interaction with resources, including digital resources, as well as the teachers' communication with each other and their students (Gueudet et al. 2014). These studies, characterizing and theorizing teaching with resources, reported significant influences on teachers' developing knowledge and practices.

Professional Development Activity Influencing Practice

Initiatives are starting to become more visible in which university teachers of mathematics explore ways in which they can develop their own teaching locally and report on outcomes. In New Zealand, the *DATUM* project (Development and Analysis of Teaching in Undergraduate Mathematics), including both mathematicians and mathematics education researchers, began as a longitudinal project to develop a model for professional development, theoretically grounded in Schoenfeld's (2010) resources, orientations, and goals (ROG) model of teacher action. Each member of the group had one of their lectures recorded and selected a short (3- to 4-min) segment for discussion, along with a brief written reflection of their ROGs. Participants were encouraged to reflect on their teaching episodes, to stimulate discussion of both mathematical and pedagogical knowledge and thereby develop their practice organically. The study has had an enduring impact on teaching practice (Barton et al. 2014).

In the UK, a professional development initiative called the *How we Teach* project consisted of a set of seminars in each of which a mathematics teacher (mathematician or mathematics educator) presented an account of some chosen aspect of their teaching which was then discussed with colleagues. Seminars were video-recorded to act as a source for others to view and analyzed to discern perspectives and issues relating to teaching and its development. The seminars were built into a university course on teaching development as an optional study for new lecturers in mathematics (Jaworski and Matthews 2011).

In a survey of research on learning and teaching mathematics at the tertiary level, Biza et al. (2016) report on an increasing interest by tertiary teachers in non-lecture pedagogies. They refer to Hayward et al. (2015) in the USA, who report on the impact on their teaching practice of a series of annual, weeklong PD workshops for college mathematics teachers on Inquiry-Based Learning (IBL) in undergraduate mathematics. Fifty-eight percentage of the teachers reported implementing IBL strategies in the year following the workshop they attended (p. 5). Biza et al. (ibid.) refer to a study of five exemplary calculus programs at US institutions in which the program had substantive, well-structured GTA (Graduate Teaching Assistant) training (Rasmussen et al. 2014).

Mathematics Support

In relation to acknowledged student difficulties with university mathematics, a network has developed over 15–20 years of providing support in mathematics for university students. This network for Mathematics Learning Support, referred to as Sigma (http://sigma-network.ac.uk/), has developed in the UK and is branching to several other countries in Europe. Support is provided one-to-one by university lecturers or GTAs who have been trained through a series of workshops covering for example methods of teaching for understanding, listening, explaining and questioning skills; individual student needs and differences. Nonmathematics skills such as counselling and empathy, dealing with mathematics anxiety and mental blocks is also included (Croft and Grove 2016; Solomon et al. 2010).

Conclusion

It is clear that much of what is reported above consists of specific activity and initiatives in known and reported areas. While this is encouraging in so far as it addresses the reported limitations, it is yet piecemeal and lacks more widespread and coherent directions internationally. It is also the case that possibly many more examples exist, not yet in the public domain, and that an international survey would be beneficial.

Cross-References

▶ Service-Courses in University Mathematics Education
▶ Teaching Practices at University Level
▶ University Mathematics Education

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Probabilistic and Statistical Thinking

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Definition

In the field of mathematics education, probabilistic thinking, statistical thinking, and probabilistic and statistical thinking are umbrella terms. Often the terms are accompanied with related terms, such as reasoning, understanding, conceptions, teaching, learning, and literacy. For example, the phrase *statistical thinking, reasoning, and literacy* is widely adopted in statistics education. Ultimately, the terms are used to identify research and practice associated with topics such as randomness, likelihood, data, chance, uncertainty, and risk. Although probability and statistics are inextricably linked, the same cannot be said for research regarding probabilistic and statistical thinking in the fields of probability and statistics education.

Probabilistic and Statistical Thinking

Formally or informally, attempts at investigating whether or not human beings are innate probabilists and/or statisticians, by and large, cement the popular notion that humans are not necessarily adept when it comes to probabilistic and statistical thinking. Certain illustrations have even achieved legendary status. In the early 1990s, for example, Marilyn vos Savant invited readers of her column "Ask Marilyn?," found in *Parade Magazine*, to submit their responses to The Monty Hall Problem (which she called the Game Show Problem). An entire book has been written about what happened next and the problem itself. Nearly 25 years later, Numberphile, which popularizes mathematics in video form, produced a video entitled Monty Hall Problem. Views of this video reached millions, but views of the problem have not changed since the 1990s. Further fortifying our arguable ineptitude with probability and statistics, experts (as they are so called) do not fare any better. Medical doctors, lawyers, teachers, and the like, too, have difficulty with basic and complex probabilistic and statistical thinking (see, e.g., the many articles and videos dedicated to scientists' attempts to explain, simply, *p-values*). Even those individuals at the top of their respective fields have demonstrated difficulty with probabilistic and statistical thinking (see, e.g., Paul Erdos' dislike for the correct solution to The Monty Hall Problem and Martin Gardner's arguable mistake with The Two Child Problem). With both experts and otherwise demonstrating difficulties with probabilistic and statistical thinking, much research, in a variety of fields, has been conducted.

Research in fields other than mathematics education (e.g., cognitive psychology) has demonstrated that merely asserting probability and

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statistics as counterintuitive only scratches the surface of what takes place during our efforts to navigate the world probabilistically and statistically. Similar research has been and is being conducted in the field of mathematics education. When combined with the recent adoption of probability and statistics in mathematics curricula worldwide, it appears that a field interested in the teaching and learning of probability and statistics *and* probabilistic and statistical thinking, such as the fields of probability and statistics education, is uniquely positioned to continue to investigate probabilistic and statistical thinking.

Probabilistic Thinking in Probability Education

As is the case with *statistical thinking*, the term *probabilistic thinking* is often accompanied with further descriptors when used in the field of probability education. Dominant terms common in the research literature include *probabilistic thinking* and *teaching and learning probability*. Lesser used terms such as *reasoning, understanding, and conceptions* are utilized and are often combined with alternative descriptors associated with probability such as *uncertainty, data, chance, randomness,* and *likelihood*. Unlike *statistics education* the term *probability education* has yet to be as deeply adopted by its respective field. Certain publications – for example, the special journal issue *Research and Development in Probability Education* (Borovcnik and Kapadia 2009) and the book *Teaching and Learning Stochastics: Advances in Probability Education Research* (Batanero and Chernoff 2018) – have started to embrace the term. However, the phrases *probabilistic thinking* and *teaching and learning probability* are still associated with recent projects, such as the book *Probabilistic Thinking: Presenting Plural Perspectives* (Chernoff and Sriraman 2014), in the field.

The field of probability education has been (and is) shaped by outside influences. In particular, research from the field of psychology has played a foundational role for research investigating probabilistic thinking in mathematics education. As Jones and Thornton (2005) detail in their overview of research into the teaching and learning of probability, "the initial research in the field was undertaken during the 1950s and 1960s by Piaget and Inhelder and by psychologists with varying theoretical orientations" (p. 66). The influence of psychologists was to such an extent that, when looking back, Jones and Thornton denoted the first chronological period of probability education research, the 1950s and 1960s, as the Piagetian Period, a period dominated by the work of Piaget and Inhelder (1975) that, to no surprise, focused on stages of development but, in doing so, revealed insights into children's thinking regarding intuition, sample space, the law of large numbers, randomization, and other probabilistic notions. The second phase, too, would be dominated by psychologists.

As Jones and Thornton continued with their overview, they noted that the second phase "was a period of prolific research on the probabilistic thinking of children and adults" (2005, p. 70). The second phase of research, denoted the Post-Piagetian Period, taking place in the 1970s and 1980s, was again ruled by psychologists: in particular, Efraim Fischbein's research on probabilistic intuitions and Amos Tversky and Daniel Kahneman's research on heuristics and biases. Fischbein's (1975) research distinguishing between *primary intuitions* and *secondary intuitions* would become foundational to further research in probability education. Probabilistic intuitions were also at the core of the *heuristics and biases program* of Tversky and Kahneman (e.g., Kahneman et al. 1982). Their *heuristic principles*, "which reduce the complex tasks of assessing probabilities and predicting values to simple judgmental operations" (Tversky and Kahneman 1974, p. 1124), would help lay the foundation for a "burgeoning growth of studies by mathematics educators" (Jones and Thornton 2005) investigating the probabilistic thinking and the teaching and learning of probability.

The field, while transitioning from the Post-Piagetian Period to the Contemporary Period, witnessed many changes. For example, mathematics educators, as opposed to psychologists, as was the case in the past, began to utilize research from the field of psychology in their own research. This shift is evidenced in the research of Shaughnessy (1977, 1981) that aimed to look at the influence of teaching on Tversky and Kahneman's heuristics principles (in particular the representativeness heuristic) and is also evidenced by mathematics educators such Konold (1989) and LeCoutre (1992) contributing their own heuristics, the outcome approach and the equiprobability bias, respectively, to the research literature. Change, at this time, was also occurring in terms of the volume of research into probabilistic thinking that was being conducted. Instead of drawing on the foundational studies from a few individuals (e.g., Piaget and Inhelder, Fischbein, and Tversky and Kahneman), the close of the second phase of research resulted in reviews of existing literature (e.g., Hawkins and Kapadia 1984). Alternatively stated, it was getting harder and harder to keep a handle on the growing number of studies investigating probabilistic thinking.

The Contemporary Period, which took place during the 1990s and 2000s, coincided with probability and statistics being adopted as a major strand of various mathematics curriculum, which resulted in "accelerated research activity into the teaching and learning of probability" (Jones and Thornton 2005, p. 79). Influenced by probability and statistics having gone to school, probability education research focused on curricula itself (e.g., when and why to introduce particular philosophical interpretations of probability), varying aged students (e.g., elementary school, middle school, high school, tertiary, and others), and different teaching and learning environments (e.g., the use of computers and simulations). In addition, research in varying probabilistic topics flourished and resulted in various threads of research in probability education. Perceptions of randomness (e.g., Batanero and Serrano 1999; Bennett 1998; Falk and Konold 1997), for example, now an established area of investigation in the field of probability education, burgeoned in the Contemporary Period. Worthy of note, in 2005, Jones and Thornton then argued that it was "premature" (p. 83) for them to historically evaluate the significance of research in the Contemporary Period. With hindsight, however, it would appear that the 1990s and 2000s would lay the foundation for various, particular threads of research in probability education (e.g., simulation, theoretical frameworks, intuition, and many others). The variety of the probability education publications published in this period also speaks to the field coming into its own.

Naturally, major publications regarding probability education existed prior to (and after) the Contemporary Period. For example, the National Council of Teachers of Mathematics (NCTM) dedicated their 1981 Yearbook to *Teaching Statistics and Probability* (Shulte and Smart 1981). The 1990s and 2000s saw publication of three major edited books: Kapadia and Borovcnik's (1991) *Chance Encounters: Probability in Education*; Jones' (2005) *Exploring Probability in School: Challenges for Teaching and Learning*; and Burrill and Elliott's (2006) editing of the 68th Yearbook of the NCTM entitled *Thinking and Reasoning with Data and Chance*. Beyond edited books, the topic of probability became a staple of various handbooks of mathematics education. For example, Borovcnik and Peard's probability chapter appeared in the (1996) *International Handbook of Mathematics Education*, Shaughnessy's chapter on probability and statistics in the (1992) *Handbook of Research on Mathematics Teaching and Learning*, and Jones, Langrall, and Mooney's probability chapter in the (2007) *Second Handbook of Research on Mathematics Teaching and Learning*. The close of the Contemporary Period would also bear witness to a special journal issue of the *International Electronic Journal of Mathematics Education*, edited by Borovcnik and Kapadia (2009), which would continue the work of the probability study group that occurred at the 11th International Congress on Mathematical Education by capturing "Research and Developments in Probability Education." This notion, that is, subsequently turning conference activity into edited books and special journal issues, would continue into the next period of research and, concurrently.

While, self-admittedly, it was a tad premature for Chernoff and Sriraman (2014) to attempt to name the period following the Contemporary Period, the manner in which they recognized particular research threads has shed some light on current and future research directions during the last (roughly) 10 years of probability education. They make reference to an attempt at a "unified approach to the teaching and learning of the classical, frequentist and subjective interpretations of probability" (p. xvii), which has been a call from various researchers for years (see, e.g., Hawkins and Kapadia 1984). Among the three dominant interpretations of probability, they note the issues inherent to the varying uses and meanings of the term "subjective probability" that remain in the literature. They foreshadow potential emergence of the topic of risk (e.g., Chernoff 2015; Prat et al. 2011) and recognize not only a renewed interest in the genericized notion of Tverksy and Kahneman's heuristics and biases program (e.g., Gilovich et al. 2002; Kahneman 2011) but also the emergence of the research of Gerd Gigerenzer (e.g., Gigerenzer et al. 1999). Ultimately, though, they settled on the Assimilation Period.

Denoting, while in the midst of, the next period of research in probability education, Chernoff and Sriraman (2014), it would appear, hedged their bet. Instead of picking and choosing a particular research thread to represent the period, they took a look at the general trend of probability education research. They noticed, as was the case in the Contemporary Period, that earlier trends associated with the field showed no signs of slowing down, but rather were increasing in volume and pace. They considered the increase of articles published in major mathematics education research journals. Also, probability specific conference activity at major mathematics education conferences (e.g., Working Groups and Topic Study Groups); and resultant special journal issues (Biehler and Pratt 2012 and Chernoff et al. 2016), and authored (Batanero et al. 2016; Batanero and Borovcnik 2016) and edited (Batanero and Chernoff 2018) books. The continuation of probability as a staple of research handbooks in education and mathematics education was also considered. Taken all together, Chernoff and Sriraman suggested that the field of probability education was becoming (further) assimilated into the field of mathematics education. Once the purview of psychologists and a handful of pioneering mathematics educators, probability education, as a field, now has all the markings of becoming a full-fledged content area in the field of mathematics education.

Statistical Thinking in Statistics Education

Unlike probability education, the term *statistics education* is deeply adopted within in the field (e.g., International Association of Statistical Education, *Statistics Education Research Journal*, *Journal of Statistics Education,* and others). *Statistical thinking*, though, is still accompanied with various terms such as understanding, learning, reasoning, teaching (and literacy). Chapter titles in the *International Handbook of Research in Statistics Education*, by way of example, have been entitled "Research on Statistics Learning and Reasoning" (Shaughnessy 2007). Thus, variations of *statistical thinking/learning/reasoning/understanding/literacy* are used as terms to help encapsulate research on *statistical thinking* in statistics education. (As will be presented, statistics education phraseology is more specifically used than in probability education.)

Compared to what is becoming known as probability education, the field of statistics education has been around for the same period of time, roughly, and has similar academic roots in the work of Piaget and Inhelder and Kahneman and Tversky. This, however, is where similarities end. The field of statistics education distinguishes itself from probability education. For example, in a chapter in the *International Handbook of Research in Statistics Education*, one that addresses what statistics education is, Zieffler, Garfield, and Fry comment: "Before we begin, we note that although probability plays an important role in statistics education, we will rarely refer to it in this chapter. We made this decision in part because we view probability as a separate discipline from statistics and only a single component of statistics education, not its entirety" (2018, p. 37). There are other differentiations for the field.

The field of statistics education distinguishes itself in terms of statistics (as opposed to mathematics) and its relation to mathematics education. As Moore (1998) notes, "statistical thinking is a general, fundamental, and independent mode of reasoning about data, variation, and chance" (p. 1257). Parsing the notions of *statistics*, *statistics education*, and *statistics education research* in the (first three chapters of the) first major section of the *International Handbook of Research in Statistics Education* supports assertions, found earlier in the said handbook, regarding "this new discipline that has come of age" (Ben-Zvi et al. 2018, p. xix). This "new subject" (Cobb 2018, p. v), one that is "grounded in science" (ibid), this "new science of statistics education" (ibid), when examined in terms of rather standard measures, does appear to have distinguished itself from other fields.

The International Association for Statistical Education (IASE), officially constituted as the education section of the International Statistical Institute (ISI), identifies itself as the international umbrella organization for statistics education. Through publications (e.g., *Statistics Education Research Journal*), conferences (e.g., International Conference On Teaching Statistics, ICOTS), and other avenues, the IASE looks to enhance statistics education across the globe. With all the markers of an independent field of study (e.g., an international association, international conferences, research journals), one that is coming of age (e.g., *International Handbook of Research in Statistics Education*), there is a bevy of research in the field despite it being relatively young.

Petocz et al. (2018) denote statistics education research as "the world of research that pertain [s] to the teaching, learning, understanding, and using of statistics and probability in diverse contexts, both formal and informal" (p. 71). Within this world of research into statistical thinking, there are agreed-upon big ideas and particular delineations. According to Shaughnessy (2007), the big ideas include understanding of centers and average, variability (in data, from data and samples, and from samples to distributions, with formal and informal inference), information garnered from samples, comparison of data sets, graph sense, and technology (see, e.g., the research of Hollylynne Stohl Lee, Rolf Biehler, Dave Pratt, Janet Ainley, and others). (It should be pointed out that research into the aforementioned topics is conducted, predominantly, with students, but investigations with teachers continue to grow as an area.) These big ideas are echoed in seven themes related to learning and understanding statistics as identified by Pfannkuch and delMas (2018): "practice of statistics, research on data, research on uncertainty, introducing children to modeling variability, learning about statistical inference, statistics learning trajectories, and research on statistics teachers' cognitive and affective characteristics" (p. 101). In a broader sense, as demonstrated in Ben-Zvi and Garfield (2004), statistics education research is delineated according to models and research frameworks from the field, which results in the major threads of *statistical literacy* (e.g., Gal 2002), *statistical thinking* (e.g., Wild and Pfannkuch 1999), and *statistical reasoning*. Whether examined in terms or topics (e.g., centers) or threads (e.g., statistical literacy), there is dominant underlying current to all statistics education research.

The Data Deluge

There are signs that our world is coming to, whether it wants to or not, fully embrace probability and statistics. For example, the amount of data the world generates is ever and more rapidly increasing. Continuing advances to technology continually increase the computing prowess sitting in our pockets. The (loosely defined) job of data scientist is consistently ranked as one of the best jobs to procure in many countries. And certain individuals (e.g., Hans Rosling, Nate Silver, Sir David Spiegelhalter, Daniel Kahneman, Arthur Benjamin, Andrew Gelman, and others) have achieved the stochastic equivalent to rock star status in popular culture. Alternatively presented, those who are able to adeptly navigate the data deluge are gaining a unique status within a world shifting its attention to data, chance, and uncertainty. Should certain trends continue, there will be a time where it behooves everyone to be adept at navigating this new world, which brings us back to the dominant underlying current to statistics education research. "Perhaps the overarching goal of statistics education is to enable students (of any age) to read, analyze, critique, and make inferences from distributions of data" (Shaughnessy 2007, p. 968). Signs that our schools are coming to fully embrace probability and statistics are starting to emerge.

Supplantation

The teaching and learning of mathematics is under attack. Public denunciations of school mathematics, for example, have been published in *The New York Times* (Andrew Hacker 2012), the *Wall Street Journal* (E. O. Wilson 2013), and *Harper's Magazine* (Nicholson Baker 2013). These public condemnations of the necessity of the teaching and learning of mathematics are nothing new. As detailed by Baker, openly questioning various aspects of the teaching and learning of mathematics (e.g., algebra) has occurred since the 1900s (e.g., William McAndrew). In the past, those who dare critique the teaching and learning of mathematics have been met with ostracism. This time, however, the situation appears slightly different.

As mentioned, we are in a data deluge. And, yes, change in education is not easy. And, yes, change in education is not quick. But in a world embracing data, chance, and uncertainty, probabilistic thinking and statistical literacy, reasoning, and thinking are becoming ever important. Students, with hundreds of other classmates, are still packing themselves into first year lecture halls all around the world. However, instead of coming to gain a grasp of first and second derivatives, students are coming to learn about quantitative reasoning, data analysis, and this new amazing job they heard of called *data scientist*.

Opportunities, then, to teach students (of any age) the big ideas of probabilistic and statistical thinking (e.g., data, center, variability, sampling, models) and to help them reason, understand, and think, about data, uncertainty, variability, and statistical inference, must be embraced–embraced not just by those in probability education, not just by those in statistics education, but also by those involved in mathematics education.

Calculus, for now, is perilously perched atop Mount School-Mathematics. But as probability and statistics move further into the mainstream and probability and statistics education moves further into the mainstream of mathematics education research, further denunciations of school mathematics may not be met with ostracism; rather they may play a pivotal role in probability and statistics education gaining prominence in elementary, secondary, and tertiary classrooms around the world and in the mathematics education research community. At the very least, the tired folklore that probability and statistics only gets taught in math class after everything else (e.g., trig, algebra, etc.) has been covered, time permitting of course, can finally be put to rest.

Cross-References

▶ Heuristics and Biases
▶ Heuristics in Mathematics Education
▶ Intuition in Mathematics Education
▶ Misconceptions and Alternative Conceptions in Mathematics Education
▶ Probability Teaching and Learning
▶ Risk Education

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Probability Teaching and Learning

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Different Meanings of Probability

While the meaning of a typical mathematical object or operation (rectangles, division, etc.) is clear and not subject to interpretation, probability has received different meanings along history that still today are challenged. Although there are no contradictions in the probability calculus per se, different philosophical theories and the emerging conceptions of probability still persist, among which the most relevant for teaching are the classical, frequentist, subjectivist, and axiomatic or formal conceptions (Batanero et al. 2005) that we briefly analyze below.

Probability reveals a dual character since its emergence: a *statistical side* was concerned with finding the objective mathematical rules behind sequences of outcomes generated by random processes through data and experiments, while another *epistemic side* views probability as a personal degree of belief (Hacking 1975).

Progress in probability was linked to games of chance; it is not surprising that the pioneer interpretation was based on an assumption of equiprobability for all possible elementary events, an assumption which is reasonable in such games as throwing dice. In the *classical definition*, given by Abraham de Moivre in 1718 in the *Doctrine of Chances* and later refined by Laplace in 1814 in his *Philosophical essay on probability*, probability is simply a fraction of the number of favorable cases to a particular event divided by the number of all cases possible in that experiment. This definition was criticized since its publication since the assumption of equiprobability of the outcomes is based on subjective judgment, and it restricts the application from the broad variety of natural phenomena to games of chance.

In his endeavor to extend the scope of probability to insurance and life-table problems, Jacob Bernoulli justified to assign probabilities to events through a frequentist estimate by elaborating the Law of Large Numbers. In the frequentist approach sustained later by von Mises or Renyi, probability is defined as the hypothetical number towards which the relative frequency tends. Such a convergence had been observed in many natural phenomena so that the frequentist approach extended the range of applications enormously. A practical drawback of this conception is that we never get the exact value of probability; its estimation varies from one repetition of the experiments (called sample) to another. Moreover, this approach is not appropriate if it is not possible to *repeat* the experiment under exactly the *same* conditions.

While in the classical and in the frequentist approaches probability is an "objective" value we assign to each event, the Bayes's theorem, published in 1763, proved that the probability for a hypothetical event or cause could be revised in light of new available data. Following this interpretation, some mathematicians like Keynes, Ramsey, or de Finetti considered probability as a personal degree of belief that depends on a person's knowledge or experience. Bayes' theorem shows that an initial (prior) distribution about an unknown probability changes by relative frequencies into a posterior distribution. Consequently, from data one can derive an interval so that the unknown probability lies within its boundaries with a predefined (high) probability. This is another proof that relative frequencies converge and justifies using data to estimate unknown probabilities. However, the status of the prior distribution in this approach was criticized as subjective, even if the impact of the prior diminishes by objective data, and de Finetti proposed a system of axioms to justify this view in 1937.

Despite the fierce discussion on the foundations, progress of probability in all sciences and sectors of life was enormous. Throughout the twentieth century, different mathematicians tried to formalize the mathematical theory of probability. Following Borel's work on set and measure theory, Kolmogorov, who corroborated the frequentist view, derived in 1933 an axiomatic. This axiomatic was accepted by the different probability schools because with some compromise the mathematics of probability (no matter the classical, frequentist or subjectivist view) may be encoded by Kolmogorov's theory; the interpretation would differ according to the school one adheres to. However, the discussion about the meanings of probability and the long history of paradoxes is still alive in intuitions of people who often conflict with the mathematical rules of probability (Borovcnik et al. 1991).

Probability in the School Curriculum

Students are surrounded by uncertainty in economic, meteorological, biological, and political settings and in their social activities such as games or sports. The ubiquity of randomness implies the student's need to understand random phenomena in order to make adequate decisions when confronted with uncertainty; this need has been recognized by educational authorities by including probability in the curricula from primary education to high school and at university level.

The philosophical controversy about the meaning of probability has also influenced teaching (Henry 1997). Before 1970, the classical view of probability based on combinatorial calculus dominated the school curriculum, an approach that was difficult, since students have problems to find the adequate combinatorial operations to solve probability problems. In the "modern mathematics" era, probability was used to illustrate the axiomatic method; however this approach was more suitable to justify theories than to solve problems. Both approaches hide the multitude of applications since the equiprobability assumption is restricted to games of chance. Consistently, many school teachers considered probability as a subsidiary part of mathematics, and either they taught it in this style or they left it out of class. Moreover, students hardly were able to apply probability in out-of-school contexts.

With increasing importance of statistics at school and progress of technology with easy access to simulation, today there is a growing interest in an experimental introduction of probability as a limit of stabilized frequencies (frequentist approach). We also observe a shift in the way probability is taught from a formula-based approach to a modern experiential introduction where the emphasis is on probabilistic experience. Students (even young children) are encouraged to perform random experiments or simulations, formulate questions or predictions about the tendency of outcomes in a series of these experiments, collect and analyze data to test their conjectures, and justify their conclusions on the basis of these data. This approach tries to show the students that probability is inseparable from statistics, and vice versa, as it is recognized in the curriculum.

Simulation and experiments can help students face their probability misconceptions by extending their experience with randomness. It is important, however, to clarify the distinction between ideally repeated situations and one-off decisions, which are also frequent or perceived as such by people. By exaggerating simulation and a frequentist interpretation in teaching, students may be confused about their differences or return to private conceptions in their decision making.

Moreover, a pure experimental approach is not sufficient in teaching probability. Though simulation is vital to improve students' probabilistic intuitions and in materialize probabilistic problems, it does not provide the key about how and why the problems are solved. This justification depends on the hypotheses and on the theoretical probability model on which the computer simulation is built, so that a genuine knowledge of probability can only be achieved through the study of some probability *theory*. However, the acquisition of such formal knowledge by students should be gradual and supported by experience with random experiments, given the complementary nature of the classic and frequentist approaches to probability. It is also important to amend these objective views with the subjectivist perspective of probability which is closer to how people think, but is hardly taken into account in the current curricula in spite of its increasing use in applications and that it may help to overcome many paradoxes, especially those linked to conditional probabilities (Borovcnik 2011).

When organizing the teaching of probability, there is moreover a need to decide what content to include at different educational levels. Heitele (1975) suggested a list of fundamental probabilistic concepts, which can be studied at various degrees of formalization, each of which increases in cognitive and linguistic complexity as one proceeds through school to university. These concepts played a key role in the history and form the base for the modern theory of probability while at the same time people frequently hold incorrect intuitions about their meaning or their application in absence of instruction. The list of fundamental concepts include the ideas of random experiment and sample space, addition and multiplication rules, independence and conditional probability, random variable and distribution, combinations and permutations, convergence, sampling, and simulation.

All these ideas appear along the curriculum, although, of course, with different levels of formalization. In primary school, an intuitive idea of probability and the ability to compute simple probabilities by applying the Laplace rule or via the estimation from relative frequencies using a simple notation seems sufficient. By the end of high school, students are expected to discriminate random and deterministic experiments, use combinatorial counting principles to describe the sample space and compute the associate probabilities in simple and compound experiments, understand conditional probability and independence, compute and interpret the expected value of discrete random variables, understand how to draw inferences about a population from random samples, and use simulations to acquire an intuitive meaning of convergence.

It is believed today that in order to become a probability literate citizen, a student should understand the use of probability in decision making (e.g., stock market or medical diagnosis) or in sampling and voting. In scientific or professional work, or at university, a more complex meaning of probability including knowledge of main probability distributions and even the central limit theorem seems appropriate.

Intuitions and Misconceptions

For teaching, it is important to take into account informal ideas that people relate to chance and probability before instruction. These ideas appear in children who acquire experience of randomness when playing chance games or by observing natural phenomena such as the weather. They use qualitative notions (probable, unlikely, feasible, etc.) to express their degrees of belief in the occurrence of random events in these settings; however their ideas are too imprecise. Young children may not see stable properties in random generators such as dice or marbles in urns and believe that such generators have a mind of their own or are controlled by outside forces.

Although older children may realize the need of assigning numbers (probabilities) to events to compare their likelihood, probabilistic reasoning rarely develops spontaneously without instruction (Fischbein 1975), and intuitions are often found to be wrong even in adults. For example, the mathematical result that a run of four consecutive heads in coin tossing has no influence on the probability that the following toss will result in heads seems counterintuitive. This belief maybe due to the confusion between hypotheses and data: when we deal with coin tossing, we usually assume that the experiment is performed *independently*. In spite of the run of four heads observed, the model still is used and, then, the probability for the next outcome remains half for heads; however intuitively these data prompt people to abandon the assumption of independence and use the pattern of past data to predict the next outcome.

Piaget and Inhelder (1951) investigated children's understanding of chance and probability and described stages in the development of probabilistic reasoning. They predicted a mature comprehension of probability at the formal operational stage (around 15 years of age), which comprises that adolescents understand the law of large numbers – the principle that explains simultaneously the global regularity and the particular variability of each randomly generated distribution. However, later research contradicted some of their results; Green's (1989) investigation with 2930 children indicates that the percentage of students recognizing random distributions decreases with age.

Moreover, research in Psychology has shown that adults tend to make erroneous judgments in their decisions in out-of-school settings even if they are experienced in probability. The well-known studies by Kahneman and his collaborators (see Kahneman et al. 1982) identify that people violate normative rules behind scientific inference and use specific *heuristics* to simplify the uncertain decision situation. According to them, such heuristics reduce the complexity of these probability tasks and are in general useful; however, under specific circumstances, heuristics cause systematic errors and are resistant to change.

For example, in the *representativeness heuristics*, people estimate the likelihood of an event taking only into account how well it represents some aspects of the parent population neglecting any other information available, no matter how relevant it is for the particular decision. People following this reasoning might believe that small samples should reflect the population distribution and consistently rely too much on them. In case of discrepancies between sample and population, they might even predict next outcomes to reestablish the alleged similarity. Other people do not understand the purpose of probabilistic methods, where it is not possible to predict an outcome with certainty but the behavior of the whole distribution, contrary to what some people expect intuitively. A detailed survey of students' intuitions, strategies, and learning at different ages may be found in the different chapters of Jones (2005) and in Jones et al. (2007).

Another fact complicates the teaching of probability (Borovcnik and Peard 1996): whereas in other branches of mathematics counterintuitive results are encountered only at higher levels of abstraction, in probability counterintuitive results abound even with basic concepts such as independence or conditional probability. Furthermore, while in logical reasoning – the usual method in mathematics – a proposition is true or false, a proposition about a random event would only be true or false after the experiment has been performed; beforehand we only can consider the *probability* of possible results. This explains that some probability theorems (e.g., the central limit theorem) are expressed in terms of probability.

Challenges in Teaching Probability

The preceding philosophical and psychological debate suggests that teachers require a specific preparation to assure their competence to teach probability. Unfortunately, even if prospective teachers have a major in mathematics, they usually studied only probability *theory* and consistently lack experience in designing investigations or simulations (Stohl 2005). They may be unfamiliar with different meanings of probability or with frequent misconceptions in their students. Research in statistics education has shown that textbooks lack to provide sufficient support to teachers: they present an all too narrow view of concepts; applications are restricted to games of chance; even definitions are occasionally incorrect or incomplete.

Moreover, teachers need training in pedagogy related to teaching probability as general principles valid for other areas of mathematics are not appropriate (Batanero et al. 2004). For example, in arithmetic or geometry elementary operations can be reversed and reversibility can be represented by concrete materials: when joining a group of three marbles with another group of four, a child always obtains the same result (seven marbles); if separating the second set from the total, the child always returns to the original set provided that the marbles are seen as equivalent (and there is hardly a dispute on such an abstraction). These experiences are vital to help children progressively abstract the structure behind the concrete situation, since they remain closely linked to concrete situations in their mathematical thinking. However, with a random experiment such as flipping a coin, a child obtains different results each time the experiment is performed, and the experiment cannot be reversed. Therefore, it is harder for children to understand

(and acknowledge) the structure behind the experiments, which may explain why they do not always develop correct probability conceptions without instruction.

Our previous discussion also suggests several important questions to be considered in future research: How should we take advantage of the multifaceted nature of probability in organizing instruction? How to conduct children to gradually view probability as an a priori degree of uncertainty, as the value to which relative frequencies tend in random experiments repeated under the same conditions, and as a personal degree of belief, where "subjectivist" does not mean arbitrariness, but use of expert knowledge? How to make older students realize that probability should be viewed as a mathematical model, and not a property of real objects? And finally, how best to educate teachers to become competent in the teaching of probability?

Cross-References

▶ Data Handling and Statistics Teaching and Learning

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Problem-Solving in Mathematics Education

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Keywords

Problem-solving · Frameworks · Digital technologies · Reasoning · Collaboration · Communication · Critical thinking

Introduction

Problem-solving approaches appear in all human endeavors. In mathematics, activities such as posing or defining problems and looking for different ways to solve them are central to the development of the discipline. In mathematics education, the systematic study of what the process of formulating and solving problems entails and the ways to structure problem-solving approaches to learn mathematics has been part of the research agenda in mathematics education. How have research and practicing problem-solving approaches changed and evolved in mathematics education, and what themes are currently investigated? Two communities have significantly contributed to the characterization and development of the research and practicing agenda in mathematical problem-solving: mathematicians who recognize that the process of formulating, representing, and solving problems is essential in the development of mathematical knowledge (Polya 1945; Hadamard 1945; Halmos 1980) and mathematics educators and teachers who are interested in understanding, explaining, and characterizing problem-solvers' cognitive, social, and affective processes that shape their ways to solve problems and to learn mathematics (Schoenfeld 1985, 1992; Lester and Kehle 1994, 2003; Lesh and Zawojewski 2007; English and Gainsburg 2016; Liljedahl and Santos-Trigo 2019). The analysis of what the development of mathematics involves and how individuals' cognitive and affective behaviors influence their problem-solving approaches provides important information for teachers to frame learning environments that aim to engage students and users of mathematics in problem-solving experiences.

Focusing of Problems in the Development of Mathematics

There are traces of mathematical problem-solving activities throughout the history of mathematics and human civilization. Arithmetic and geometric problems appear in Babylonian clay tablets, the Greeks' three classical geometric problems (squaring the circle, trisecting an angle, and doubling a cube), and the Hilbert (1902) list of 23 mathematics problems illustrates trends of the discipline in different times. Devlin (2002) introduces, to a wide audience, seven mathematical problems, known as the Millennium Problems, that were proposed by the Clay Mathematics Institute in 2000 and were considered as the most significant unsolved problems of contemporary mathematics.

Mathematical problems address and inform on themes and contents studied at different times, and also the attempts to find their solutions contribute directly to the development of new areas and solution methods. Making explicit how mathematicians pose and solve problems has been also an issue of interest within the mathematics community.

Hadamard (1945) published *Essay on the Psychology of Invention in the Mathematics Field* in which he asked 100 physicists and mathematicians to describe how they worked and solved their field problems. As a result, Hadamard proposed a four-step model that resembles features of the Gestalt psychology (Wertheimer 1945) that describes experts' problem-solving approaches in terms of four phases: preparation, incubation, illumination, and verification. Similarly, Polya (1945) wrote on his own experience to work and do mathematics. Based on retrospection (looking back at events that already have taken place) and introspection methods (self-examination of one's conscious thought and feelings), he explains what the process of solving mathematical problems involves. To this end, he proposed a general framework that describes four problem-solving stages (understanding the problem, devising a plan, carrying out the plan, looking back). Polya recognizes that solving problems is a practical skill that students develop or learn by observing and imitating how teachers or people solve problems and by doing problems directly. In this process, he suggests that teachers should ask questions to guide their students throughout all phases and students should pose and pursue questions as a means to identify and activate resources and strategies to solve problems. Polya also illustrates and discusses the use and power of several heuristics (analogy, drawing figures or auxiliary constructions, special cases, etc.) to represent, explore, and solve different problems. Polya's work has been seminal in mathematics education and has inspired the design and implementation of research programs in mathematical problem-solving. Halmos (1980) pointed out that mathematics consists of axioms, theorems, proofs, definitions, methods, etc. and all are essential ingredients; but "what mathematics *really* consists of is problems and solutions" (p. 519).

In mathematical instruction at the university level, the Moore method (https://en.wikipedia.org/wiki/Moore_method) to learn advanced mathematics involves providing students a list of definitions, theorems, and course problems that students are asked to understand, explain, and prove within a learning community that fosters the members' participation including the instructor as a moderator (Halmos 1994). The mathematics community's explicit recognition of the importance of problems in the making and development of the discipline and its intents to unveil and explain what solving problems entails provides foundations to think of ways to study and foster the students' process to learn mathematics as a set of structured problem-solving activities.

Mathematics Education and Problem-Solving Developments

Research developments in mathematics education go hand in hand with the conceptual frameworks, research designs, and methods used to delve into learners thinking. The most salient feature of the problem-solving research agenda is that the themes, questions, and research methods have changed perceptibly and significantly through time. Shifts in research themes are intimately related to shifts in research designs and methodologies (Lester and Kehle 2003).

Early problem-solving research relied on quantitative methods and statistical hypothesis testing designs; later, approaches were, and continue to be, based mostly on qualitative methodologies. Krutetskii (1976) relied on set of mathematical tasks to analyze and characterize the mathematical abilities of gifted children. Krutetskii' study not only provides a robust characterization of the mathematical abilities of these children but also illustrates ways to elicit their thinking through the use of a variety of mathematical problems. The interest in qualitative studies that aim to examine in detail the process that subjects or students show in understanding mathematical knowledge and developing problem-solving competencies led mathematics educators to design and implement research programs to investigate teachers/students' problem-solving behaviors.

Research programs structured around problem-solving have made significant contributions to the understanding of the complexity involved in developing the students' deep comprehension of mathematics ideas, in using research results in the design and structure of curricular frameworks, and in directing mathematical school practices.

Schoenfeld (1985) implemented a research program that focused on analyzing students' development of mathematical ways of thinking that reflects a microcosm or features of experts' mathematical practices. A key issue in his program was to characterize what it means to think mathematically and to document how students become successful, or develop proficiency in solving mathematical tasks. He used a set of nonroutine tasks to engage first year university students in problem-solving activities that explicitly included the implementation of heuristic strategies in solving the problems. As a result, Schoenfeld proposed a framework to explain and document students' problem-solving behaviors in terms of four dimensions or categories: the use of basic mathematical resources or knowledge base, the use of cognitive or heuristic strategies, the use of metacognitive or self-monitoring and control strategies, and students' beliefs about mathematics and problem-solving. These categories are intertwined and shed light on ways to orient the gradual students' development of problem-solving competencies.

Schoenfeld's framework has been used extensively not only to document the extent to which problem-solvers succeed or fail in their problem-solving attempts but also to organize and foster students' development of problem-solving experiences in the classrooms. Schoenfeld (1992) also reported on the strengths and limitations associated with the use of Polya's heuristics.

"Polya's characterization did not provide the amount of detail that would enable people who were not already familiar with the strategies to be able to implement them" (Schoenfeld 1992, p. 353). That is, students need to work on ways to identify or break down a general heuristic into a collection of sub-strategies and analyze their conditions under which they can be applied or used in different domains (algebra, geometry, calculus, etc.). Similarly, Perkins and Simmons (1988) present a model to characterize what they call deep understanding of a domain (mathematics, science, or programming) in terms of four interrelated frames:

The *content frame* that includes definitions, facts, algorithms, rules, or operations associated with the subject matter and strategies for monitoring the activation of these elements

The *problem-solving frame* that refers to the domain's problem-solving strategies including ways to monitor problem-solvers' own solution process and beliefs about problem-solving

The *epistemic frame* that includes ways to reason and validate domain results

The *inquiry frame* that involves strategies to understand and develop domain contents

Schoenfeld (2015) updated his 1985 problem-solving framework to explain how and why problem-solvers make decisions that shape and guide their problem-solving behaviors. He proposes three constructs to explain in detail what problem-solvers do on a moment-by-moment basis while engaging in a problem-solving approach: the problem-solver's resources, goals, and orientations. He suggests that these constructs offer teachers, and problem-solvers in other domains, tools for reflecting on their practicing decisions. Schoenfeld uses this framework to analyze and predict the behaviors of mathematics and science teachers and a medical doctor.

A salient feature in these frameworks is the importance for problem-solvers to engage in metacognitive behaviors to regulate or monitor their own process to make decisions and to solve problems.

Curriculum Proposals and Instruction

The NCTM (1989) launched a curriculum framework structured around problem-solving approaches. This framework was updated in 2009 (NCTM 2000, 2009) and conceptualizes a problem-solving approach as a way of fostering mathematical reasoning and sensemaking activities. Throughout the proposal, there are different examples in which reasoning and sensemaking activities are interwoven. Phases such as analyzing a problem or concept, implementing a strategy, looking for connections, and reflecting on a solution are discussed in terms of reasoning habits (finding key concepts, seeking for patterns, considering special cases, examining the meaning of procedures and operations, looking for connections, interpreting solutions, examining different problem-solving approaches, generalizing solutions, etc.) that students need to internalize and practice during the solution process.

Recently, the Common Core State Mathematics Standards (CCSMS) (2010) also identified problem-solving as one of the standard processes to develop students' mathematical proficiency. Through all grades, students are encouraged to engage in problem-solving practices that involve making sense of problems, and persevere in solving them, to look for and express regularity in repeated reasoning, to use appropriate tools strategically, etc. In terms of instruction, mathematical tasks and ways to discuss them within the learning environments are important elements to implement problem-solving activities.

The importance for students to work on nonroutine problems was shown in Selden et al. (1989) study; they reported that even students, who had passed their calculus course, experienced serious difficulties to identify and activate concepts needed to solve the problems. Thus, problem-solving instruction should foster and encourage students to work on nonroutine tasks in which they have an opportunity to always look for different ways to represent, explore, and solve mathematical problems and reflect on the extent to which their solution methods can be applied to solve other problems (Santos-Trigo 2007).

Some problem-solving approaches rely on promoting scaffolding activities to gradually guide students' construction of problem-solving abilities. Instructional strategies involve fostering and valuing students' small group participation, plenary group discussions, the instructor presentations through modeling problem-solving behaviors, and the students' constant mathematical reflection. Lesh and Zawojewski (2007) identify modeling activities as essential for students to develop knowledge and problem-solving experiences. They contend that in modeling processes, interactive cycles represent opportunities for learners to constantly reflect on, revise, and refine tasks' models. Thus, the multiplicity of interpretations of problem-solving has become part of the identity of the field.

Regional Problem-Solving Developments and the Use of Digital Tools

Regional or country mathematics education traditions also play a significant role in shaping and pursuing a problem-solving agenda. Artigue and Houdement (2007) summarized the use of problem-solving in mathematics education in France in terms of two influential and prominent theoretical and practical frameworks in didactic research: the theory of didactic situations (TDS) and the anthropological theory of didactics (ATD). They also pointed out that in the French compulsory education, curriculum proposals recognize solving problems as the source and goal to mathematical learning. In the Netherlands, the problem-solving approach is associated with the theory of Realistic Mathematics that pays special attention to the process involved in modeling the real-world situations. They also recognized a strong connection between mathematics as an educational subject and problem-solving as defined by the PISA program (Doorman et al. 2007).

Cai and Nie (2007) pointed out that problem-solving activities in Chinese mathematics education have a long history and are viewed as a goal to achieve and as an instructional approach supported more on experience than a cognitive analysis. In the classroom teachers stress problem-solving situations that involve discussion: *one problem multiple solutions*, *multiple problems one solution*, and *one problem multiple changes*. "The purpose of teaching problem solving in the classroom is to develop students' problem-solving skills, help them acquire ways of thinking, form habits of persistence, and build their confidence in dealing with unfamiliar situations" (Cai and Nie 2007, p. 471).

Digital Technologies and Mathematical Problem-Solving

Significant developments and use of digital technologies, such as smartphones or tablets, are transforming not only the ways in which people communicate or interact with others but are also providing new opportunities for teachers and students to represent, explore, and solve mathematical problems and to extend mathematical discussions beyond formal settings (Santos-Trigo and Reyes-Martínez 2018).

There are *mathematical action technologies* that can be used to represent, explore, and work on mathematical tasks (Dynamic Geometry Systems (DGS), computational and representational tools (Wolfram|alpha), MicroWorlds, or computer simulations) and *conveyance technologies* (Dick and Hollebrands 2011) that are useful to explain, share, and discuss mathematical ideas or problems (communication applications such as Skype or FaceTime and presentation technologies such as Keynote or PowerPoint). There are also online platforms that include videos to explain mathematical themes, examples of problems and proposed assignments (https://www.khanacademy.org/coach/dashboard), or online developments (https://www.wikipedia.org) in which students can consult information about contents, concepts, or events. All these technology developments provide different affordances for teachers and students to work on mathematical problems, and the goal is that students can use them throughout all their learning experiences.

DGS' affordances allow teachers and students to represent or model concepts and mathematical problems dynamically. Then, within this model, they can orderly move objects and observe the behavior of some object attributes and identify some invariance, patterns, or possible mathematical relationships among those objects. In terms of the problem understanding phase, the use of a DGS provides opportunities for students to pose questions regarding ways to select the tool affordances for representing or reconstructing concepts or figures that appear in problem statements. Similarly, during the exploration phase, it becomes important to quantify object attributes (lengths, angles, slopes, areas, perimeters, etc.) and visualize or trace their graphic behaviors. Thus, finding loci of points or objects that emerge when particular points are moved within a model and using sliders to explore particular parameter behaviors are powerful strategies to identify some object mathematical properties and to solve problems. Likewise, students can rely on DGS affordances to extend (by varying values or dimensions of object representations) the initial problem domain and explore generalization of the solution methods. Communication technologies are also important for teachers and students to continue or extend mathematical discussions beyond formal settings. That is, students can share their ideas, questions, or comments via a digital wall (Padlet), and other group peers can follow the discussion and pose others' questions.

English and Gainsburg (2016) emphasize the importance of connecting problem-solving activities with the demands of modern life and work. To this end, the coordinated use of digital technologies opens new paths for people to participate in development and practice of the four twenty-first-century key competencies: critical thinking and problem-solving, communication, collaboration, and creativity and innovation. Thus, the formulation of problems, finding always different solutions paths, presenting and sharing ideas and results, and reflecting on ways to apply solution methods to solve other problems are problem-solving activities to foster teachers and students' development of these four competencies.

The systematic use of technology not only enhances what teachers and students do with the use of paper and pencil but also extends and opens new routes and ways of reasoning for students and teachers to develop mathematics knowledge (Hoyles and Lagrange 2010; Santos-Trigo and Reyes-Rodriguez 2016). Thus, emerging reasoning associated with the use of the tools needs to be characterized and made explicit in curriculum and conceptual frameworks in order for teachers to incorporate it and to foster its development in teaching practices.

In terms of curriculum materials and instruction, the use of several digital technologies could transform the rigid and often static nature of the content presentations into a dynamic and flexible format where learners can access to and rely on several digital developments (dynamic software, online encyclopedias, widgets, videos, etc.) during their solution of mathematical tasks.

The advent and use of digital technology in society and education influence and shape the academic problem-solving agenda. The learners' tools appropriation to use them in problem-solving activities involves extending previous frameworks and to develop different methods to explain mathematical processes that are now enhanced with the use of those tools.

Directions for Future Research

In a retrospective account, research in problem-solving has generated not only interesting ideas and useful results to frame and discuss paths for students to develop mathematical knowledge and problem-solving proficiency; it has also generated ways to incorporate this approach into the design of curriculum proposals and instructional approaches.

Recent digital developments are shaping and influencing how people and students deal with problem-solving activities. Mathematics teacher education and teachers professional development programs need to include ways for prospective and practicing teachers to incorporate the coordinated use of diverse technologies in their teaching practices. Specifically, they need to analyze and reflect on what both mathematical action and conveyance types of technologies bring to curriculum contents and students' problem-solving competencies and to instruction.

In this context, teachers, together with researchers, need to be engaged in problem-solving experiences where all have an opportunity to discuss and design problem-solving activities and ways to implement and evaluate them in and beyond actual classroom settings. In addition, there are different paths for students to develop mathematical thinking, and the use of tools shapes the ways they think of, represent, and explore mathematical tasks or problems. Then, theoretical frameworks used to explain learners' construction of mathematical knowledge need to capture or take into account the different ways of reasoning that students might develop as a result of using a set of tools during the learning experiences. As a consequence, there is a need to develop or adjust current problem-solving frameworks to account not only the students processes of appropriation of the tools but also the need to characterize the ways of reasoning, including the use of new heuristics, for example, dragging in dynamic representations, with which students construct learning as a result of using digital tools in problem-solving approaches. In addition, it is important to develop methodological tools to observe, analyze, and evaluate achievements and behaviors of problem-solver groups that involve the use of digital technology.

Cross-References

- ▶ Critical thinking in Mathematics Education
- ▶ Inquiry-Based Mathematics Education
- ▶ Learning Practices in Digital Environments
- ▶ Mathematical Ability
- ▶ Mathematical Modelling and Applications in Education
- ▶ Metacognition
- ▶ Realistic Mathematics Education
- ▶ Scaffolding in Mathematics Education
- ▶ Task-based Interviews in Mathematics Education
- ▶ Technology and Curricula in Mathematics Education
- ▶ Visualization and Learning in Mathematics Education

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Professional Learning Communities in Mathematics Education

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Characteristics

During the past decade, professional learning communities have drawn the attention of educationists interested in school leadership, school learning, and teacher development. Professional learning communities aim to establish school cultures, which are conducive to ongoing learning and development, of students, teachers, and schools as organizations (Stoll et al. 2006). Professional learning communities refer to groups of teachers collaborating to inquire into their teaching practices and their students' learning with the aim of improving both. In order to improve practice and learning, professional learning communities interrogate their current practices and explore alternatives in order to refresh and re-invigorate practice (McLaughlin and Talbert 2008). Exploring alternatives is particularly important in mathematics education where a key goal of teacher development is to support teachers' orientations towards understanding and engaging students' mathematical thinking in order to develop conceptual understandings of mathematics among students.

A key principle underlying professional learning communities is that if schools are to be intellectually engaging places, all members of the school community should be intellectually engaged in learning on an ongoing basis (Curry 2008). Professional learning communities are "fundamentally about learning – learning for pupils as well as learning for teachers, learning for leaders, and learning for schools" (Katz and Earl 2010, p. 28). Successful learning communities are those that challenge their members to reconsider taken-for-granted assumptions in order to generate change, for example, challenging the notion that working through procedures automatically promotes conceptual understandings of mathematics. At the same time, not all current practices are problematic, and successful professional learning communities integrate the best of current practice with ideas for new practices.

A number of characteristics of successful professional learning communities have been identified: they create productive relationships through care, trust, and challenge; they de-privatize practice and ease the isolation often experienced by teachers; they foster collaboration, interdependence, and collective responsibility for teacher and student learning; and they engage in rigorous, systematic enquiry on a challenging and intellectually engaging focus. Professional

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learning communities in mathematics education focus on supporting teachers to develop their own mathematical knowledge and their mathematical knowledge for teaching, particularly in relation to student thinking (Brodie 2011; Curry 2008; Jaworski 2008; Katz et al. 2009; Little 1990).

The notion of collective learning in professional learning communities is important. The idea is that teachers who work together learn together, making for longer-term sustainability of new practices and promoting community-generated shifts in practice, which are likely to provide learners with more coherent experiences across the subject or school (Horn 2005; McLaughlin and Talbert 2008). Professional learning communities support teachers to "coalesce around a shared vision of what counts for high-quality teaching and learning and begin to take collective responsibility for the students they teach" (Louis and Marks 1998, p. 535). Ultimately, a school-wide culture of collaboration can be promoted, although working across subject disciplines can distract from a focus on subject knowledge (Curry 2008). Networked learning communities, where professional learning communities come together across schools in networks, provide further support and sustainability for individual communities and improved teacher practices (Katz and Earl 2010).

There is differing terminology for learning communities, which illuminate subtle but important differences in how communities are constituted. These include "communities of practice," "communities of enquiry," and "critical friends groups." The key emphasis in the notion of *professional* learning is that it signals the focus of the community and the learning as both data-informed and knowledge-based.

Data-Informed and Knowledge-Based Enquiry

Professional learning communities can be established within or across subjects, and in each case the communities would choose different focuses to work on. Working within mathematics suggests that the focus would be on knowledge of and intellectual engagement with mathematics and the teaching and learning of mathematics. Effective communities focus on addressing student needs through a focus on student achievement and student work, joint lesson and curriculum planning, and joint observations and reflection on practice, through watching actual classroom lessons or videotaped recordings of classroom practice. Mathematics learning communities support teachers to focus on learner thinking through examples of learners solving rich problems (Borko et al. 2008; Whitcomb et al. 2009) or through teachers' analyzing learner errors (Brodie 2011).

In many cases data comes from national tests, and teachers work together to understand the data that the tests present and to think about ways to improve their practice that the data suggests. Working with data as a mechanism to improve test scores can be seen as a regulatory practice, with external accountability to school managers and education department officials. Proponents of teacher-empowered professional learning communities argue strongly that the goal of such data analysis must be to inform teachers' conversations in the communities, as a form of internal accountability to knowledge and learning (Earl and Katz 2006). Data can also include teachers' own tests, interviews with learners, learners' work, and classroom observations or videotapes.

The professional focus of professional learning communities requires that the learning in these communities be supported by a knowledge base as well as by data. As teachers engage with data, their emerging ideas are brought into contact with more general findings from research. Jackson and Temperley (2008) argue for a model where practitioner knowledge of the subject, learners, and the local context meets public knowledge, which is knowledge from research and best practice. The interaction between data from classrooms and wider public knowledge is central in creating professional knowledge, for two reasons. First, without outside ideas coming into the communities' conversations, they can become solipsistic and self-preserving and may continue to maintain the status quo rather than invigorate practice. Second, data and knowledge work together to promote internal accountability, to the learners and teachers and to support the creation of new professional knowledge, which is research-based, locally relevant, and collectively generated. (Data-informed practice is different from evidence-based practice. Evidence-based practice suggests that only research-based evidence is good enough to inform teacher professional development. Data-informed professional development suggests that teachers themselves, with some expert guidance, can and should interpret data that is available to them and integrate research knowledge with their local circumstances).

Leadership

Leadership in professional learning communities is central, particularly in helping to bring together data from practice and the findings of research. Leaders can be school-based or external, for example, district officials or teacher-educators from universities. For long-term sustainability, there should be leadership within the school, or within a cluster of schools.

Two key roles have been established as important for leaders in professional learning communities. The first is promoting a culture of inquiry and mutual respect, trust, and care, where teachers are able to work together to understand challenges in their schools more deeply and support each other in the specific challenges that they face as teachers. The second is to support teachers to focus on their students' knowledge and subsequently their own knowledge and teaching practices. The second role is crucial in supporting professional learning communities where subject-specific depth is the goal, depth in learning and knowledge for both teachers and learners.

It is important for leaders in professional learning communities to also be learners and to be able to admit their own weaknesses (Brodie 2011; Katz et al. 2009). At the same time, it is important for leaders to have and present expertise, which helps the community to move forward. In mathematics, leaders need to recognize opportunities for developing mathematical knowledge and knowledge of learning and teaching mathematics among teachers, for example, what counts as appropriate mathematical explanations, representations, and justifications and how these can be communicated with learners. Other functions for leaders in professional learning communities are developing teachers' capacities to analyze classroom data; supporting teachers to observe and interpret data rather than evaluate and judge practice; supporting teachers to choose appropriate problem of practices to work on, once the data has been interpreted; and helping teachers to work on improving their practice and monitoring their own and progress in doing this, as well as their learners' progress (Boudett and Steele 2007). So leadership in professional learning communities is a highly specialized task.

Impact and Research

There is a growing body of research that shows that professional learning communities do promote improved teacher practices and improved student achievement (Stoll et al. 2006). However, the evidence is mixed depending on which aspects of learning different studies choose to focus on. Research into professional learning communities invariably must confront how to recognize and describe learning, both in the conversations of the community and in classrooms. It is well known from situated theory that learning does not travel untransformed between sites, rather it is recontextualized and transformed as it travels from classrooms to communities and back again.

A second issue that research into professional learning communities must confront is the relationship between group and individual learning. While the focus of the community is on group learning and interdependence, ultimately each person contributes in particular ways to the community and brings particular expertise, and different people will learn and grow in different ways. Kazemi and Hubbard (2008) suggest a situated framework for research into how the individual and the group coevolve in mathematics professional learning communities. Group and individual trajectories can be examined in relationship to each other, through a focus on particular practices and artifacts of practice discussed by the community. How particular practices travel from the classroom into the community and back again can be traced through linking what happens in the community to what happens in teachers' classrooms.

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Psychological Approaches in Mathematics Education

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Keywords

Behaviorism · Collaborative learning · Constructivism · Concept development · Teaching experiments · Design research · Technology-Based learning environments · Abstraction · Socio-mathematical norms

Characteristics

Cognitive psychology, developmental psychology, and educational psychology are general fields of research for which mathematics education naturally seems one among many domains of application. However, the history of these domains of research and of the development of research in mathematics education is much more complex, and not at all hierarchical. For example, in their monumental *Human Problem Solving* (Newell and Simon 1972), Newell and Simon acknowledged that many of their ideas (which became among the fundamentals of Cognitive Psychology) were largely inspired from George Pólya's *How to Solve It* (Pólya 1945). Another prestigious link is of course Piaget's *Epistémologie Génétique* – his theory of human development: the theory was based on memorable experiments in which Piaget designed conservation tasks in which mathematical entities were focused on (number, quantity, length, proportions, etc.). Also Cole's Cultural Psychology (1996) is largely based on the comparison between mathematical practices in different societies. The reasons for these ties are profound, and beyond the very different approaches adopted, mathematics represents a domain through which human cognition, cognitive development, or human development can be studied. We focus here on some psychological approaches adopted in mathematics education. Although these approaches have come out at different times, approaches were not merely replaced and each of them is still vibrant in the community of researchers in mathematics education.

The Constructivist Approach

Our review of psychological approaches in mathematics is not exhaustive. We mention the approaches that contribute to our understanding of learning and teaching processes and that can help in what we consider as their improvement. For this reason, we overlooked behavioristic approaches. We will begin with constructivism – a learning theory with a very long history that can be traced to John Dewey. The simple and general idea according to which learning occurs when humans actively engage in tasks has been understood very differently by different psychologists. For some, constructivism means discovery-based teaching techniques, while for others, it means self-directedness and creativity. Wertsch (1998) adopts a social version of constructivism – socioculturalism – to encourage the learner to arrive at his or her version of the truth, influenced by his or her background, culture, or embedded worldview. Historical developments and symbol systems, such as language, logic, and mathematical systems, are inherited by the learner as a member of a particular culture, and these are learned throughout the learner's life. The fuzziness and generality of the definition of constructivism led to inconsistent results. It also led to the memorable "math wars" controversy in the United States that followed the implementation of constructivist-inspired curricula in schools with textbooks based on new standards. In spite of many shortcomings, the constructivist approach had the merit to lead scientists to consider the educational implications of the theories of human development of Piaget and Vygotsky in particular in mathematics education (von Glasersfeld 1989; Cobb and Bauersfeld 1995).

The Piagetian Approach: Research on Conceptions and Conceptual Change

The impact of Piaget's theory of human development had and still has an immense impact on research on mathematics education. Many researchers adapted the Piagetian stages of cognitive growth to describe learning in school mathematics. Collis' research on formal operations and his notion of closure (Collis 1975) are examples of this adaptation. With the multi-base blocks (also known as Dienes blocks), Dienes (1971) was also inspired by Piaget's general idea that knowledge and abilities are organized around experience to sow the seeds of contemporary uses of manipulative materials in mathematics instruction to teach structures to young students.

Since the 1970s researchers in science education realized that students bring to learning tasks alternative frameworks or misconceptions that are robust and difficult to extinguish. The idea of misconception echoed Piagetian ideas according to which children consistently elaborate understandings of reality that do not fit scientific standards. Researchers in mathematics education adopted these ideas in terms of tacit models (Fischbein 1989) or of students' concept images (Tall and Vinner 1981). These frameworks were seen as theories to be replaced by the accepted, correct scientific views. Bringing these insights into the playground of learning and development was a natural step achieved through the idea of conceptual change. This idea is used to characterize the kind of learning required when new information comes in conflict with the learners' prior knowledge usually acquired on the basis of everyday experiences. It is claimed that then a major reorganization of prior knowledge is required – a conceptual change. The phenomenon of conceptual change was first identified for scientific concepts and then in mathematics (e.g., the acquisition of the concept of fraction requires radical changes in the preexisting concept of natural number, Hartnett and Gelman 1998). Misconceptions were thought to develop when new information is simply added to the incompatible knowledge base, producing synthetic models, like the belief that fractions are always smaller than the unit. Learning tasks, in which students were faced with a cognitive conflict, were expected to replace their misconceptions by the current accepted conception. Researchers in mathematics education continue studying the discordances and conflicts between many advanced mathematical concepts and naïve mathematics. Intuitive beliefs may be the cause of students' systematic errors (Fischbein 1987; Stavy and Tirosh 2000; Verschaffel and De Corte 1993). Incompatibility between prior knowledge and incoming information is one source of students' difficulties in understanding algebra (Kieran 1992), fractions (Hartnett and Gelman 1998), and rational numbers (Merenluoto and Lehtinen 2002). The conceptual change approach is still vivid because of its instructional implications that help to identify concepts in mathematics that are going to cause students great difficulty, to predict and explain students' systematic errors, to understand how counterintuitive mathematical concepts emerge, to find the appropriate bridging analogies, and more generally, to develop students as intentional learners with metacognitive skills required to overcome the barriers imposed by their prior knowledge (Schoenfeld 2002). However, harsh critiques pointed out that cognitive conflict is not an effective instructional strategy and that instruction that "confronts misconceptions with a view to replacing them is misguided and unlikely to succeed" (Smith et al. 1993, p. 153). As a consequence, misconceptions research in mathematics education was abandoned in the early 1990s. Rather, researchers began studying the knowledge acquisition process in greater detail or as stated by Smith et al. (1993) to focus on "detailed descriptions of the evolution of knowledge systems" (p. 154) over long periods of time.

Departing from Piaget: From Research on Concept Formation to Teaching Experiments

The fine-grained description of knowledge systems in mathematics education was initiated as an effort to adapt his theory to mathematics education (Skemp 1971). Theories of learning in mathematics were elaborated, among them the theory of conceptual fields (Vergnaud 1983), the notion of tool-object dialectic (Douady 1984, 1986), and theories of process-object duality of mathematical conceptions (Sfard 1991; Dubinsky 1991). Van Hiele's theory of development of geometric thinking (Van Hiele 2004) seems at a first glance to fit Piaget's view of development with its clear stages. However, it clearly departed from Piaget's theory in the sense that changes result from teaching rather than from independent construction on the part of the learner. The method of the teaching experiment was introduced to map trajectories in the development of students' mathematical conceptions. Steffe et al. (2000) produced fine-grained models of students' evolving conceptions that included particular types of interactions with a teacher and other students. It showed that learning to think mathematically is all but a linear process, but that what can be seen as mistakes or confusions may be essential in the learning process. Moreover, "misconceptions" often resist teacher's efforts, but they eventually are necessary building blocks in the learning of conceptions. In the same vein, Schwarz et al. (2009) elaborated the RBC model of abstraction in context to identify the building blocks of mathematical abstraction which are often incomplete or flawed. Such studies invite considering alternative approaches to understand the development of mathematical thinking. The RBC model takes into account the impressive development of sociocultural approaches in mathematics education.

Sociocultural Approaches

Descriptions of students learning in teaching experiments stressed the importance of the social plane – of the interactions between teacher and students. Vygotsky's theory of human development was a natural source of inspiration for researchers in mathematics education in this context. A series of seminal studies on street mathematics (e.g., Nunes et al. 1993) on the ways unschooled children used mathematical practices showed the situational character of mathematical activity. Rogoff's (1990) integration of Piagetian and Vygotskian theories to see in guided participation a central tenet of human development fitted these developments in research in mathematics education. Rogoff considered learning and development as changes of practice. For her, learning is mutual as the more knowledgeable (the teacher) as well as students learn to attune their actions to each other. Cobb and colleagues took the mathematics classroom in its complexity as the natural context for learning mathematics (Cobb et al. 2001; Yackel and Cobb 1996). He introduced the fundamental notion of social and socio-mathematical norms to point at constructs that result from the recurring enactment of practices in classrooms (an embryonic version of this notion had already been elaborated by Bauersfeld (1988)). Cobb and colleagues showed that those norms are fundamental for studying individual and group learning: learning as a change of practice entails identifying the establishment of various norms. Vygotsky's *intersubjectivity* as the necessary condition for maintaining communication was replaced by Cobb and colleagues by *taken-as-shared beliefs*. Cobb also considered the mathematical practices of the classrooms (standards of mathematical argumentation, ways to reasoning with tools and symbols) as other general collective constructs to be taken into account to trace learning. Norms are constructed in the mini-culture of the classroom in which researchers are not only observers but actively participate in the establishment of this mini-culture. Cobb adopts here a new theoretical approach in the Learning Sciences – Design Research (Collins et al. 2004). This interesting approach led to many studies in mathematics education, but also raised the tough issue of generalizability of design experiments.

Although according to Cobb and his followers, learning is highly situational, knowledge that emerges in the classroom is presented in a decontextualized form that fits (or not) accepted mathematical constructs. The writings of influential thinkers challenged this view. In *L'archéologie du savoir*, Michel Foucault (1969) convincingly traced the senses given to ideas such as "madness" along the history through the analysis of texts. Instead of identifying knowledge as a static entity, he forcefully claimed that human knowledge should be viewed as "a kind of discourse" – a special form of multimodal communication. Leading mathematics education researchers adopted this perspective (Lerman 2001; Kieran et al. 2002). In her theory of commognition, Sfard (2008) viewed discourse as what changes in the process of learning, and not the internal mental state of an individual learner. From this perspective, studying mathematics learning means exploring processes of discourse development. The methodology of the theory of commognition relies on meticulous procedures of data collecting and analysis. The methods of analysis are adaptations of techniques developed by applied linguists or by discursively oriented social scientists. The discourse of the more knowledgeable other is for Sfard indispensable, not only as an ancillary help for the discovering student but as a discourse to which he or she should persist to participate, in spite of the fact its nature is incommensurable with the nature of his or her own discourse. Sfard's theory and Cobb's theory, which stemmed from research in mathematics education, have become influential in the Learning Sciences in general.

Open Issues

Leading modern thinkers such as Bakhtin have headed towards *dialogism*, a philosophy based on dialogue as a symmetric and ethical relation between agents. This philosophical development has yielded new pedagogies that belong to what is called *Dialogic Teaching*, and new practices, for example, (un-)guided small group collaborative and argumentative practices, or teacher's facilitation of group work. A good example of dialogic teaching enacted in mathematics classrooms is *Accountable Talk* (Michaels et al. 2009). Dialogic Teaching raises harsh psychological issues as in contrast with sociocultural approaches for which adult guidance directs emergent learning, dialogism involves symmetric relations.

Numerous technological tools have been designed by CSCL (Computer-Supported Collaborative Learning) scientists to facilitate (un-)guided collaborative work for learning mathematics. These new tools enable new discourse practices with different synchronies and enriched blended multimodalities (oral, chat, computer-mediated actions, gestures). Virtual Math Teams (Stahl 2012) is a representative project which integrates powerful dynamic mathematics applications such as GeoGebra in a multiuser platform for (un) guided group work on math problems, so that small groups of students can share their mathematical explorations and co-construct geometric figures online. In a recent book, *Translating Euclid*, Stahl (2013) convincingly shows how collaborating students can reinvent Euclidean geometry with minimal guidance and suitable CSCL tools. The possibilities opened by new technologies challenge the tenets of sociocultural psychology: the fact that students can collaborate during long periods without adult guidance challenges neo-Vygotskian approaches for which adult guidance is central for development. To what extent can it be said that the designed tools embody adult discourse? In spite of the fact that the teacher is often absent, new forms of participation of the teacher fit dialogism (e.g., moderation as caring but minimally intrusive guidance). The psychological perspective that fit changes in participation and the role of multiple artifacts in these changes is an extension of the Activity Theory, the theory of Expansive Learning (Engeström 1987) to the learning of organizations rather than the learning of individuals. The mechanisms of the emergent learning of the group are still mysterious, though. It seems then, that, again, mathematics education pushes psychology of learning to unconquered lands.

Cross-References

- ▶ Abstraction in Mathematics Education
- ▶ Activity Theory in Mathematics Education
- ▶ Argumentation in Mathematics Education
- ▶ Collaborative Learning in Mathematics Education
- ▶ Concept Development in Mathematics Education
- ▶ Constructivism in Mathematics Education
- ▶ Constructivist Teaching Experiment
- ▶ Design Research in Mathematics Education
- ▶ Information and Communication Technology (ICT) Affordances in Mathematics Education
- ▶ Learning Environments in Mathematics Education
- ▶ Sociomathematical Norms in Mathematics Education

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