The Second Handbook of Research on the Psychology of Mathematics Education

The Journey Continues

Ángel Gutiérrez, Gilah C. Leder and
Paolo Boero (Eds.)

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To the young researchers, throughout the world,
who are the future of mathematics education research
and of the PME community

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FOREWORD

Another Decade of PME Research

PME stands for Psychology of Mathematics Education; it is a society whose members form the International Group for the Psychology of Mathematics Education (IGPME). It holds an international conference annually, hosted by PME members in a diversity of countries around the world. From its beginnings in 1976, Psychology has underpinned the mathematics education research reported at PME conferences. For example, studies of students' learning of mathematical topics (e.g., ratio, algebra, calculus, geometry…) and recognition of the difficulties certain topics present to students. However, it has long been understood that the P (for Psychology) embraces a range of human sciences, such as (for example) Philosophy, Sociology, Anthropology and Semiotics, which have become central to the research in Mathematics Education of quite a few PME members. While the central themes of research continue to be the learning and teaching of mathematics, and cognitive studies root PME firmly within the Psychological domain, we have seen focuses on other themes like constructivism, socio-cultural theories, linguistics, equity and social justice, affect, and on the professional lives of teachers emerging over the years.

In all cases, research published in PME proceedings has gone through a critical review process. Reviewing is undertaken by PME members who have attended several PME conferences and had their own research reports in previous proceedings. PME is open to researchers in mathematics education throughout the world, and reviewers reflect the cultural and geographic diversity of PME itself. The review process requires that papers accepted for publication focus clearly on aspects of mathematics education and satisfy a set of criteria with demands on theory, methodology, reporting of results and discussion of implications and impact of the research. It is indicative of the quality of PME research reports and conference proceedings that PME papers are respected alongside those in high quality research journals.

In 2006, a PME handbook celebrating 30 years of PME was produced as a milestone for the PME community (1976–2006). This handbook synthesised PME research over the 30-year period and demonstrated the developing themes mentioned above through its chapters. Authors were chosen to acknowledge the scientific quality of their research and their active contributions to PME conferences over the years and, as a whole, to celebrate the diversity of PME culturally and geographically.

We now present a new handbook celebrating another decade of PME research – 40 years. The 40th PME conference is held in Szeged, Hungary. This is particularly fitting since Hungary was the birthplace of George Polya, who has been a great inspiration to PME members and students of mathematics widely over the years. His seminal book "How to Solve It" has influenced the doing of mathematics through problem-solving and the use of heuristics of problem solving. In fact, the title of Polya's book is taken as the title for PME 40, and as a theme for the conference. The choice of guest speaker, Alan Schoenfeld, reflects the theme: Alan having been one of the pioneers in mathematical problem solving building on Polya's work.

The editors of this new handbook were invited by the PME International Committee to produce a volume celebrating the most recent decade of PME. Their work in producing the handbook started with a survey of research published in PME proceedings since 2006, the recognition of key themes in this work and invitation to authors to study and provide a synthesis of each of the themes. Their introduction provides details of this process and a rationale for the themes chosen, showing a diversity from focuses within mathematics itself towards key aspects of the learning and teaching of mathematics and its relations to society and culture. Themes include numbers, algebra, geometry, functions and calculus, proof and argumentation, problem solving, mathematical modelling, language, the use of digital technology, curriculum and assessment, teachers' knowledge and professional development, affect, and the socio-cultural-political axis in understanding mathematics education.

PME as a society is alive and well. Recent conferences in Taipei, Kiel, Vancouver, and Hobart have been extremely well attended. In recent years, PME has introduced a special day for early career researchers (the ERD) before each annual conference. These days have also been well attended, and participants have then attended the main conference. This means that PME is actively encouraging a new generation of researchers with every conference. In the coming years we plan to have conferences in Singapore (2017) and Sweden (2018) and are in conversation with other nations for planning conferences after this.

I recommend this handbook to all researchers in Mathematics Education. You will find here a strong taste of the research in PME, a synthesis of recent research and indications for future research directions. My thanks go to the editors and all authors and reviewers for their contribution to this important work.

Barbara Jaworski
President of PME
On behalf of the PME International Committee (IC)
February 2016

INTRODUCTION

A handbook compiling the research produced by the PME Group from its very beginning until 2005 was published in 2006 to celebrate the first 30 years of existence of the PME Group. During the last ten years the activities of the PME Group have undoubtedly grown and diversified. From inspection of the more recent conference presentations and Proceedings it is readily apparent that research areas have continued to evolve. It thus seems an appropriate moment to release a new handbook which captures both the new directions that have emerged as well as providing a rich overview of areas which continue having a sustained record of explorations by the PME community. The second PME handbook is published to celebrate the 40 years of activity of the PME Group. It focuses primarily on the research activities over the last ten years (2006–2015) and can be seen as a ready sequel to the first PME handbook, which covered the period 1976–2005. The proximity of the timing of the 2015 PME conference and a critical deadline to ensure the timely publication of this handbook has led to a slightly lighter review in several chapters of papers included in the 2015 Proceedings.

To test our impression that changes have occurred, since 2005, in the research interests of the PME Group, we analyzed the indexes of the Proceedings, identified the presentations related to the various research topics, and compared the different tallies. As for the first handbook, the editors' most sensitive decision was to use this analysis to select the topics for the chapters of the new handbook. The result was a list of fourteen chapters which cover the core and most relevant parts of the activity of the PME Group during the last ten years.

By comparing the indexes of the two handbooks, the main differences which have resulted from the evolution of researchers' interests can readily be seen. The emergence of new research directions was already becoming evident during the first years of this century. Since many of these explorations were still in an initial phase of development, it was deemed premature to include them in the first handbook. However, some of these research areas (like language or modelling) have increased in momentum and relevance, and now certainly warrant inclusion in the second handbook. Main "traditional" research areas (like algebra, arithmetic, geometry, and calculus) as well as domains that were "new" ten years ago (socio-cultural, political, affectivity, ICTs, and teacher related issues) have retained a dominant presence in the Proceedings. Accordingly they form part of the second handbook – although, in some cases, with reduced coverage.

Once we had decided which topics to include in the handbook, our second important task was to identify authors who could take on the responsibility for writing the chapters. We believed that it was important to select researchers who had not contributed a chapter to the first handbook and considered that effective coverage of each field would be enhanced by the selection of a team of at least two authors, located, for most chapters, in different parts of the world. Finally, to achieve optimum continuity between the two handbooks, we asked, whenever possible, for an author from the first handbook to act as a reviewer of the pertinent chapter in the second handbook. Each chapter, it should be noted, was constructively reviewed by at least two PME members, with the selection of reviewers based on their expertise in the relevant area. For this group, too, we aimed at geographic diversity. In summary, the 31 authors in this handbook came from 15 countries (or 17 countries if we count their place of birth); the 27 reviewers from 13 different countries. The combination of the scientific quality of authors and reviewers, and their wide geographical distribution, have given voice to diverse approaches, perspectives and delivered a meaningful document of relevance to both mature and emerging researchers.

The handbook chapters are organized into the same four sections used in the first handbook. The first group of chapters correspond to topics related to mathematics content areas: algebra, arithmetic, geometry (including measurement and visualization), and calculus. The second section, the main one in terms of page volume, is devoted to transverse topics: proof, ICTs, language, curriculum and assessment, concept learning, problem solving, and modelling. The third group of chapters comprises those focused on social, cultural, political or affective aspects of teaching and learning of mathematics. Finally, the last but equally important section consists of a chapter devoted to pre- and in-service teachers' activity.

In closing, we acknowledge the efforts of all those committed to the continuing growth and evolution of the PME Group, the persistent search for new knowledge aimed at fostering teachers', students', and society's understanding and appreciation of mathematics and its productive application in their personal and professional lives. Our thanks are also extended to the members of the International Committee who serve the Group and take care of scientific, organizational, and administrative matters that require attention if the health of the PME Group is to be preserved, to the local organizers of the annual PME conferences, who selflessly strive to provide PME members with the best possible environment to celebrate the yearly meeting, and, above all, to the PME members with their shared aim of achieving a better mathematical education for all members of society.

Ángel Gutiérrez
Gilah C. Leder
Paolo Boero

REVIEWERS OF THE CHAPTERS IN THIS VOLUME

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Rudolf Strässer (Germany)
Michal Tabach (Israel)
Lieven Verschaffel (Belgium)

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The following persons acted as Presidents of the PME Group:

Efraim Fischbein (Israel)	1977–1980
Richard R. Skemp (UK)	1980–1982
Gérard Vergnaud (France)	1982–1984
Kevin F. Collis (Australia)	1984–1986
Pearla Nesher (Israel)	1986–1988
Nicolas Balacheff (France)	1988–1990
Kathleen M. Hart (UK)	1990–1992
Carolyn Kieran (Canada)	1992–1995
Stephen Lerman (UK)	1995–1998
Gilah C. Leder (Australia)	1998–2001
Rina Hershkowitz (Israel)	2001–2004
Chris Breen (South Africa)	2004–2007
Fou-Lai Lin (Taiwan)	2007–2010
João Filipe Matos (Portugal)	2010–2013
Barbara Jaworski (UK)	2013–2016

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The following persons acted as Chairs of the Local Organizing Committees of the PME International Conferences:

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PME 2	1978	Elmar Cohors-Fresenborg	Osnabrück (Germany)
PME 3	1979	David Tall	Warwick (UK)
PME 4	1980	Robert Karplus	Berkeley (USA)
PME 5	1981	Claude Comiti	Grenoble (France)
PME 6	1982	Alfred Vermandel	Antwerpen (Belgium)
PME 7	1983	Rina Hershkowitz	Shoresh (Israel)
PME 8	1984	Beth Southwell	Sidney (Australia)
PME 9	1985	Leen Streefland	Noordwijkerhout (The Netherlands)
PME 10	1986	Leone Burton and Celia Hoyles	London (UK)
PME 11	1987	Jacques C. Bergeron	Montreal (Canada)
PME 12	1988	Andrea Borbás	Veszprem (Hungary)
PME 13	1989	Gérard Vergnaud	Paris (France)
PME 14	1990	Teresa Navarro de Mendicuti	Oaxtepex (Mexico)
PME 15	1991	Paolo Boero	Assisi (Italy)
PME 16	1992	William E. Geeslin	Durham (USA)
PME 17	1993	Nobuhiko Nohda	Tsukuba (Japan)
PME 18	1994	João Pedro da Ponte	Lisbon (Portugal)
PME 19	1995	Luciano Meira	Recife (Brazil)
PME 20	1996	Angel Gutiérrez	Valencia (Spain)
PME 21	1997	Erkki Pehkonen	Lahti (Finland)
PME 22	1998	Alwyn Olivier	Stellenbosch (South Africa)
PME 23	1999	Orit Zaslavsky	Haifa (Israel)
PME 24	2000	Tadao Nakahara	Hiroshima (Japan)

LOCAL ORGANIZERS OF THE PME INTERNATIONAL CONFERENCES

No.	*Year*	*Chair*	*Place*
PME 25	2001	Marja van den Heuvel-Panhuizen	Utrecht (The Netherlands)
PME 26	2002	Anne D. Cockburn	Norwich (UK)
PME 27	2003	A. J. (Sandy) Dawson	Honolulu (USA)
PME 28	2004	Marit J. Høines	Bergen (Norway)
PME 29	2005	Helen L. Chick	Melbourne (Australia)
PME 30	2006	Jarmila Novotná	Prague (Czech Republic)
PME 31	2007	Woo Jeong-Ho	Seoul (Korea)
PME 32	2008	Olimpia Figueras	Mexico D.F. (Mexico)
PME 33	2009	Marianna Tzekaki	Thessaloniki (Greece)
PME 34	2010	Márcia Maria Fusaro Pinto	Belo Horizonte (Brazil)
PME 35	2011	Behiye Ubuz	Ankara (Turkey)
PME 36	2012	Tai-Yih Tso	Taipei (Taiwan)
PME 37	2013	Aiso Heinze	Kiel (Germany)
PME 38	2014	Cynthia Nicol and Peter Liljedahl	Vancouver (Canada)
PME 39	2015	Kim Beswick	Hobart (Australia)
PME 36	2016	Csaba Csíkos	Szeged (Hungary)

EDITORS OF THE PROCEEDINGS OF THE PME INTERNATIONAL CONFERENCES

Due to the large number of citations in the chapters of this handbook to papers in PME Proceedings, and to conform with space limitations, the editors decided that a shortened format would be used for references to the PME Proceedings. To acknowledge the editorship of the PME Proceedings, a full listing is provided below:

No.	*Year*	*Editors*
PME 1	1977	There were no proceedings published.
PME 2	1978	E. Cohors-Fresenborg & I. Wachsmuth
PME 3	1979	D. Tall
PME 4	1980	R. Karplus
PME 5	1981	Equipe de Recherche Pédagogique
PME 6	1982	A. Vermandel
PME 7	1983	R. Hershkowitz
PME 8	1984	B. Southwell, R. Eyland, M. Cooper, J. Conroy, & K. Collis
PME 9	1985	L. Streefland
PME 10	1986	Univ. of London Institute of Education
PME 11	1987	J. C. Bergeron, N. Herscovics, & C. Kieran
PME 12	1988	A. Borbás
PME 13	1989	G. Vergnaud, J. Rogalski, & M. Artigue
PME 14	1990	G. Booker, P. Coob, & T. Navarro de Mendicuti
PME 15	1991	F. Furinghetti
PME 16	1992	W. Geslin & K. Graham
PME 17	1993	I. Hirabayashi, N. Nohda, K. Shigematsu, & F.-L. Lin
PME 18	1994	J. P. da Ponte & J. F. Matos
PME 19	1995	L. Meira & D. Carraher
PME 20	1996	L. Puig & A. Gutiérrez
PME 21	1997	E. Pehkonen
PME 22	1998	A. Olivier & K. Newstead

No.	*Year*	*Editors*
PME 23	1999	O. Zaslavsky
PME 24	2000	T. Nakahara & M. Koyama
PME 25	2001	M. van den Heuvel-Panhuizen
PME 26	2002	A. D. Cockburn & E. Nardi
PME 27	2003	N. A. Pateman, B. J. Dougherty, & J. T. Zilliox
PME 28	2004	M. J. Høines & A. B. Fuglestad
PME 29	2005	H. L. Chick & J. L. Vincent
PME 30	2006	J. Novotná, H. Moraová, M. Krátká, & N. Stehlíková
PME 31	2007	J.-H. Woo, H.-C. Lew, K.-S. Park, & D.-Y. Seo
PME 32	2008	O. Figueras, J. L. Cortina, S. Alatorre, T. Rojano, & A. Sepúlveda
PME 33	2009	M. Tzekaki, M. Kaldrimidou, & H. Sakoni
PME 34	2010	M. M. F. Pinto & T. F. Kawasaki
PME 35	2011	B. Ubuz
PME 36	2012	T.-Y. Tso
PME 37	2013	A. M. Lindmeier & A. Heinze
PME 38	2014	P. Liljedahl, C. Nicol, S. Oesterle, & D. Allan
PME 39	2015	K. Beswick, T. Muir, & J. Wells
PME 36	2016	C. Csíkos, A. Rausch, & J. Szitányi

PME INTERNATIONAL GROUP CONFERENCES

The Proceedings of the PME International Conferences are available to the PME members at the PME web page http://www.igpme.org/ (except for PME4, PME5, PME6, PME8, and PME10). Many PME proceedings can be freely retrieved from the ERIC web page http://www.eric.ed.gov/. The table below indicates the ERIC ED numbers for all the Proceedings of the PME International Conferences stored at ERIC.

No.	*Year*	*Place*	*ERIC number*
2	1978	Osnabrück, Germany	ED226945 (not available)
3	1979	Warwick, United Kingdom	ED226956 (not available)
4	1980	Berkeley, California, USA	ED250186 (not available)
5	1981	Grenoble, France	ED225809 (not available)
6	1982	Antwerp, Belgium	ED226943 (not available)
7	1983	Shoresh, Israel	ED241295 (not available)
8	1984	Sydney, Australia	ED306127 (not available)
9	1985	Noordwijkerhout, The Netherlands	ED411130, ED411131
10	1986	London, United Kingdom	ED287715 (not available)
11	1987	Montréal, Canada	ED383532
12	1988	Veszprém, Hungary	ED411128, ED411129
13	1989	Paris, France	ED411140, ED411141, ED411142
14	1990	Oaxtepex, Mexico	ED411137, ED411138, ED411139
15	1991	Assisi, Italy	ED413162, ED413l63, ED413164
16	1992	Durham, New Hampshire, USA	ED383538
17	1993	Tsukuba, Japan	ED383536
18	1994	Lisbon, Portugal	ED383537
19	1995	Recife, Brazil	ED411134, ED411135, ED411136
20	1996	Valencia, Spain	ED453070, ED453071, ED453072, ED453073, ED453074
21	1997	Lahti, Finland	ED416082, ED416083, ED4l6084, ED416085

No.	*Year*	*Place*	*ERIC number*
22	1998	Stellenbosch, South Africa	ED427969, ED427970, ED427971, ED427972
23	1999	Haifa, Israel	ED436403
24	2000	Hiroshima, Japan	ED452301, ED452302, ED452303, ED452304
25	2001	Utrecht, The Netherlands	ED466950
26	2002	Norwich, United Kingdom	ED476065
27	2003	Honolulu, Hawaii, USA	ED500857, ED500859, ED500858, ED500860
28	2004	Bergen, Norway	ED489178, ED489632, ED489538, ED489597
29	2005	Melbourne, Australia	ED496845, ED496859, ED496848, ED496851
30	2006	Prague, Czech Republic	ED496931, ED496932, ED496933, ED496934, ED496939
31	2007	Seoul, Korea	ED499419, ED499417, ED499416, ED499418

PME NORTH AMERICAN CHAPTER (PME-NA) CONFERENCES

Many Proceedings of the PME-NA Conferences can be freely retrieved from the ERIC web page http://www.eric.ed.gov/ The table below indicates the ERIC ED numbers for all the Proceedings of the PME-NA Conferences published separately from the Proceedings of PME International Conferences stored at ERIC.

No.	*Year*	*Place*	*ERIC number*
3	1981	Minneapolis, Minnesota, USA	ED223449
4	1982	Athens, Georgia, USA	ED226957 (not available)
5	1983	Montreal, Quebec, Canada	ED289688
6	1984	Madison, Wisconsin, USA	ED253432 (not available)
7	1985	Columbus, Ohio, USA	ED411127
8	1986	East Lansing, Michigan, USA	ED301443 (not available)
10	1988	Dekalb, Illinois, USA	ED411126
11	1989	New Brunswick, New Jersey, USA	ED411132, ED411133
13	1991	Blacksburg, Virginia, USA	ED352274
15	1993	Pacific Grove, Califomia, USA	ED372917
16	1994	Baton Rouge, Louisiana, USA	ED383533, ED383534
17	1995	Columbus, Ohio, USA	ED389534
18	1996	Panama City, Florida, USA	ED400178
19	1997	Bloomington-Normal, Illinois, USA	ED420494, ED420495
20	1998	Raleigh, North Carolina, USA	ED430775, ED430776
21	1999	Cuernavaca, Morelos, Mexico	ED433998
22	2000	Tucson, Arizona, USA	ED446945
23	2001	Snowbird, Utah, USA	ED476613
24	2002	Athens, Georgia, USA	ED471747

PART 1

COGNITIVE ASPECTS OF LEARNING AND TEACHING CONTENT AREAS

FERNANDO HITT AND ALEJANDRO S. GONZÁLEZ-MARTÍN

1. GENERALIZATION, COVARIATION, FUNCTIONS, AND CALCULUS

1. INTRODUCTION

In this chapter we review the main contributions of PME to research on the topics of functions and calculus, identifying ongoing trends as well as newly emerging issues and approaches. The first part of this chapter (Section 2) refers mainly to the chapter on *advanced mathematical thinking* (Harel, Selden, & Selden, 2006) in the last *Handbook of Research on the Psychology of Mathematics Education* (Gutiérrez & Boero, 2006), which is where most of the PME research results concerning functions and calculus appear. However, the field has evolved, and a considerable part of this chapter is devoted to identifying issues we consider to be some of the major approaches and research topics that have emerged in recent years.

Our chapter examines research that has been carried out on the topics of functions and calculus. At first glance, this would appear to be a 'condensed' area of research, however the reality is very different from what we first imagined. Research on the teaching and learning of functions extends to the early grades, and *Early algebra* researchers advocate encouraging algebraic thinking beginning in primary school, using a functional approach. This led us to consider some PME papers that look at this *Early algebra* perspective, allowing us to explore the origins of functional thinking in primary school by examining the main contributions of PME to the perspective of the functional approach. Furthermore, the topic of functions is taught beginning in secondary school, leading to the introduction of calculus and its study at the university level. In the last ten years, research conducted at the university level has also evolved both in terms of approaches and research topics (Artigue, Batanero, & Kent, 2007; Nardi, Biza, González-Martín, Gueudet, & Winsløw, 2014; Rasmussen, Marrongelle, & Borba, 2014), adapting to the characteristics of a rather different and varied educational level. As a consequence, when we were asked to write a chapter about 'Functions and Calculus,' we were actually confronted with a wide range of topics and even some contradictions. For instance, there is a long tradition of research on teacher training for primary and secondary education, but scarce research focused on the university level; there is abundant research on teaching practices in primary and secondary education, but a major gap appears at the tertiary level; other discrepancies are covered later in this chapter. It would

Á. Gutiérrez, G. C. Leder & P. Boero (Eds.), The Second Handbook of Research on the Psychology of Mathematics Education, 3–38.
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be impossible to parse this diversity of research and multitude of approaches from primary school to the university level, and in this chapter we instead identify those works that, in our view, have propelled knowledge on the topic in the last ten years. Our choices are, of course, influenced by our own experience as researchers and the account presented here reflects our personal vision.

Concerning the theoretical approaches used to conduct research on the topics of functions and calculus, ten years ago, Harel et al. (2006) highlighted the fact that theoretical frameworks used in studies on *advanced mathematical thinking* (AMT) were largely cognitive. This phenomenon was not restricted to the PME community, but was a popular trend in research on advanced levels, which for a long time concentrated on "identifying cognitive processes underlying the learning of mathematics at advanced levels, investigating the relationships of these processes with respect to those at play at more elementary levels, and understanding students' difficulties with advanced mathematical concepts" (Artigue et al., 2007, p. 1011). While this situation has changed (more quickly in research on the primary and secondary levels than the tertiary level), the PME proceedings still publish a number of papers that follow some of these cognitive approaches, such as the *concept image – concept definition* approach or the use of representations. Although PME and PME-NA have played an important role in the consolidation of AMT, research in the last years has been critical of this approach and some of its implicit ideas, as Artigue et al. (2007) summarize. These critiques may have been the reason certain terms appear less frequently (or, at least, are being used more judiciously). For example, the Congress of European Research in Mathematics Education (CERME) had an *Advanced mathematical thinking* working group until its sixth edition (2009), but shifted to the *University mathematics education* working group in 2011 (Nardi, González-Martín, Gueudet, Iannone, & Winsløw, 2011), with some of the AMT content being redistributed to other groups. Consequently, we will not refer to AMT in this chapter and will instead refer to specific content related to functions and calculus and, of course, to the theoretical approaches pertaining to problems of teaching and learning this content.

As mentioned above, until ten years ago, researchers were using mainly cognitive approaches, especially in studying higher levels of education. Since then, some of these approaches have undergone developments that have opened up wider perspectives. For instance, research focusing on the cognitive (for example, articulation among representations, Duval, 1999) has evolved to other types of research connected to sociocultural processes, where communication in the classroom is the key ingredient (Mariotti, 2012; Radford, 2003, 2009). This last type of research has been at the origin of some interesting task-design activities (e.g. Prusak, Hershkowitz, & Schwarz, 2013). This is just one example, and later in this chapter we will discuss other instances of theoretical developments that have emerged in the last ten years. Another major evolution in the field has seen different approaches being used in a coordinated way; this has sparked an interest within the

PME community in comparing theories (e.g. Boero et al., 2002; Presmeg, 2006a) and in examining the networking of theories, as evidenced at the 2010 (Bikner-Ahsbahs et al., 2010), and 2014 (Clark-Wilson et al., 2014) Research Forums.

It is worth noting that PME and PME-NA communities have been extremely prolific in developing research related to the topics of functions and calculus. It would be therefore impossible to present a comprehensive summary, and in this chapter we address what we consider to be the most important advances, alongside research conducted outside the PME community. Although we will mainly refer to papers published in PME proceedings during the last ten years, references to the evolution of research that has appeared in journals are inevitable, and some of the papers presented during PME conferences have led to, or come from, papers published in international journals.

This chapter is divided into five main sections (not including this introduction). Section 2 synthetizes the main PME contributions developed using cognitive approaches, in most cases following pre-existing theoretical perspectives (although we also include some new approaches). The abundant literature on functions in different school grades led us to create a whole section summarizing the main results on this topic, which are discussed in Section 3, with a particular focus on the transition from mental to semiotic representations. In Section 4 we examine some attempts to expand past purely cognitive approaches, drawing mainly on socio-cultural or institutional perspectives and on the networking of theories. Section 5 addresses topics of research that have received more attention in recent years, compared to the previous handbook. Finally, in the last section, we reflect on the main contributions of PME research with respect to functions and calculus, and explore avenues for future research.

2. PME CONTRIBUTIONS USING COGNITIVE APPROACHES

2.1. Concept Image and Concept Definition

One important change becomes apparent when comparing the PME production of the last ten years with that of previous decades: the marked decline in the number of papers following the *concept image – concept definition* approach. This may be due to the shift in research towards favoring semiotic representations and issues that are more social and cultural rather than solely cognitive, as discussed in the introduction. Another possible reason is the trend towards using other cognitive approaches, as we will discuss later in this section.

The term *concept image* was introduced to define “the total cognitive structure that is associated with [a given] concept, which includes all mental pictures and associated properties and processes” (Tall & Vinner, 1981, p. 152). On the other hand, *concept definition* describes “a form of words used to specify that concept” (p. 152). This approach has been useful to show, for instance, that a learner can hold a contradictory concept image and concept definition. Building on the gap between the concept definition and the concept image, as well as the relationships between intuitive and formal knowledge (as considered by Fischbein, 1999), Kidron and Picard (2006) constructed an activity based on the discrete-continuous interplay to help university students understand the notion of limit in the definition of the derivative. Using a model from the field of dynamical systems – the logistic equation – as well as Euler's method to see the effect of replacing $\lim_{\Delta x \to 0} \Delta x / \Delta y$ by $\Delta x / \Delta y$, their data reveal the existence of the 'treasured intuition' that "*gradual causes have gradual effects* and that *small changes in a cause should produce small changes in its effect*" (p. 443, emphasis in the original). Although their activities helped some students overcome treasured intuitions and enrich their concept image, the treasured intuitions seemed to be quite persistent. They also pointed at the link between different representations as a reason for students' success at overcoming the gap. We will return to the importance of representations in Sections 3.1 and 3.3. This work was pursued by Kidron (2009), using Fischbein's notion of mental model (Fischbein, 2001), to show that "the role of intuitive structures does not come to an end when analytical (formal) forms of thinking become possible" (p. 314), again demonstrating the tenacity of certain treasured intuitions. In particular, Kidron's work shows that tacit models can coexist with logical reasoning, even in advanced students, and underlines the fact that some pictorial models may emerge when facing abstract questions, again reinforcing the importance of using different representations.

Although use of this approach in PME has dropped off significantly in recent years, it is not a static approach, as Artigue et al. (2007) clearly show. For instance, Bingolbali and Monaghan (2008) proposed a reinterpretation of the *concept image* construct, criticizing the fact that most studies using it adopt a purely cognitive approach. Their proposition – in connection with Bingolbali, Monaghan and Roper, 2006, discussed in Section 5.5 – takes into account the learning context, particularly in undergraduate studies. They demonstrate how department affiliations can have an impact on students' development of concept images (their work focuses on derivatives), which are influenced by teaching practices and departmental perspectives. Although their work adds an institutional component to the *concept image – concept definition* approach (we discuss the advantages of institutional approaches in Section 4.1), the impact of their work on PME research is unclear. However, it is worth noting that ten years ago, Harel et al. (2006) posed the question, "is a learner's current concept image a consequence of a specific teaching approach or is it an unavoidable construct due to the structure and limitation of the human brain, mind, culture, and social interaction?" (p. 163). Although institutional approaches (see Section 4.1) seem to provide answers to this question, the recent reinterpretation of the *concept image* construct also appears to respond to it, and it will be interesting to see the new directions this approach might take in light of this development.

2.2. The Duality between Process and Object

Like the *concept image – concept definition* approach, the APOS (Action, Process, Object, Scheme) framework has been less present in PME in the last 10 years. The main tenets of this approach, derived from Piaget's ideas about reflective abstraction, are clearly presented by Artigue et al. (2007), so we will only cite an example concerning infinity using an innovative perspective and developed over a number of years.

Mamolo (2014) used APOS theory as a lens through which to interpret her participants' struggle with various questions and paradoxes concerning infinity, in looking at the differences between *potential infinity* and *actual infinity* – both quite present in many notions and procedures of calculus; the former seen as a process in which every moment in time is finite, but also goes on forever, and the latter seen as a completed entity that envelops that which was previously potential (Fischbein, 2001). These two notions can be associated with the process and object conceptions of infinity, respectively, the latter being the encapsulation of the former. In this perspective, "through encapsulation, the infinity becomes cognitively attainable" (Dubinsky, Weller, McDonald, & Brown, 2005, p. 346) and can be conceived as an object, a complete entity which can be acted upon. However, recent studies have shown that in some cases, de-encapsulating infinity back to a process seems to be a useful strategy to cope with infinity (Brown, McDonald, & Weller, 2010).

Taking an innovative approach and studying paradox resolution, and building on previous work (Mamolo & Zazkis, 2008), Mamolo (2014) approached the participants' understanding of 'acting' on transfinite cardinal numbers via arithmetic operations, focusing particularly on struggles with the indeterminacy of transfinite subtractions. She used the Ping-Pong Ball Conundrum (P1, see Mamolo & Zazkis, 2008) with a variant (P2):

> P1 – Imagine and infinite set of ping-pong balls numbered 1, 2, 3…, and a very large barrel; you will embark on an experiment that will last for exactly 60 seconds. In the first 30s, you will place balls 1–10 into the barrel and then remove ball 1. In half the remaining time, you place balls 11–20 into the barrel, and remove ball 2. Next, in half the remaining time (and working more quickly), you place balls 21–30 into the barrel, and remove ball 3. You continue this task ad infinitum. At the end of the 60s, how many balls remain in the barrel? (p. 169)
>
> P2 – Rather than removing the balls in order, at the first time interval remove ball 1; at the second time interval, remove ball 11; at the third time interval, remove ball 21; and so on… At the end of this experiment, how many balls remain in the barrel? (p. 170)

Mamolo studied the work of two subjects, a high-achieving fourth year Mathematics major who had received formal instruction on comparing infinite sets, and a university lecturer who taught prospective teachers mathematics and didactics (including comparing cardinalities of infinite sets). While the first participant was able to cope with P2, the second one could not, shifting his attention from describing cardinalities of sets to enumerating their elements, and reasoning informally rather than deductively, in what is described as "attempts to make use of properties of a process of *infinitely many finite entities* rather that make use of properties of an object of *one infinite entity*" (p. 175, emphasis in the original). The behavior of the two participants led Mamolo (2014) to identify two ways of 'acting on infinity': (1) through the use of coordinating sets with their cardinalities and using bijections between sets, and, (2) through de-encapsulation of the object of infinite set to extend properties of finite cardinals to the transfinite case. Her results point to tensions between object, process, and de-encapsulation of an object that warrant further research, which could shed light on the uses of potential and actual infinity in questions dealing with calculus.

2.3. Embodied Cognition

The growing contribution to research of the theories of embodied cognition has already been acknowledged (e.g. Artigue et al., 2007), however, their use in advanced mathematics remains rare. Embodied cognition sees mathematical ideas as grounded in sensory-motor experience (Lakoff & Núñez, 2000) and considers the centrality of learners' gestures in grasping mathematical ideas. Two main conjectures of this approach are that mathematical abstractions grow to a large extent out of bodily activities (i.e. the latter are a part of conceptualizing processes), and that understanding and thinking are perceptuo-motor activities that are distributed across different areas of perception and motor action (Nemirovsky, 2003).

Before 2005, this approach was used in calculus in PME, for instance, by Maschietto (2004), who studied a key issue of the introduction of calculus – the global/local game – with the help of graphic-symbolic calculators. Her main hypothesis was that the zoom-controls of the calculator could support the production of gestures and metaphors that could help students shift from a global to a local point of view, this being seen as a major aspect of transition in calculus. To tackle this issue, she designed a didactical engineering[1] sensible to the principles of embodied cognition. Her paper explored the relationship between the physical features of the calculator (specifically, the different zooms) and the bodily activity involved. Her results seem to indicate that the exploration of several functions through the zooming process, aiming to shift between the local and the global, was supported by gestures and language, and that these remained in the students' repertoire even when the calculators were not available, which agrees with Hähkiöniemi's (2008) results concerning derivatives – except for the use of calculators – which we discuss in Section 3.3.

In 2005, Nemirovsky and Rasmussen (2005) – in connection with their work in Rasmussen, Nemirovsky, Olszervski, Dost and Johnson (2004) – also used this approach for an even more advanced mathematical topic: systems of differential equations. They constructed a physical tool called a 'water wheel,' based too on the premise of the "rich connections between kinesthetic activity and how people qualitatively understand and interpret graphs of motion" (p. 9). Interested in the notion of transfer, they explored how prior kinesthetic experiences with a physical tool (the water wheel) can provide students with resources that can be generalized to work with symbolic equations (in their case, systems of differential equations). Their results show how the students' interaction with the water wheel helped them develop sensitivity to the roles of different variables in a system of differential equations, as well as to the connections among them. One important implication of this work concerns the known dissociation of symbolic and graphical aspects of calculus concepts for students. Nemirovsky and Rasmussen argue that a bodily interpretation, or feeling, of the meaning of many topics in calculus could help students relate the results of calculations to the motion they describe, and hence their graphic representation.

More recently, Swidan and Yerushalmy (2013) also used this approach, taking into account elements of the objectification theory (which considers learning to be a process of becoming aware of the knowledge that exists in the culture, Radford, 2003), and the important role of accumulation in building the notion of integral (see Thompson & Silverman, 2008). In the three cases explored in this section, evidence shows a strong bodily connection with the tools used, becoming a 'bridge' between abstract mathematical concepts and students. However, the lack of studies on this approach at advanced levels calls for caution, and more research needs to be developed to better understand the knowledge that students build through this type of activity (Artigue et al., 2007, p. 1024). In any case, it seems that the mediations of the teacher in this kind of activity play a crucial role, particularly in helping students associate their developed knowledge with the targeted institutional forms of knowledge.

2.4. Other Approaches

Other approaches have been used in research on the topics of functions and calculus. For instance, the conceptual change approach, which postulates the necessity of going through intermediate states before gaining an understanding of mathematical notions. Using this approach, Vamvakoussi, Christou and Van Dooren (2010) showed the main difficulties students have apprehending the density property in different sets of numbers (rational, irrational and real numbers), and particularly the strong impact of the nature of interval end points (e.g. natural, decimal, or rational numbers) on students' answers related to the amount of numbers in between; these difficulties may interfere with the learning of notions such as convergence or the epsilon-delta definition of limit. Their results agree with those of Pehkonen, Hannula, Maijala and Soro (2006), who developed a longitudinal study of students in grades 5 to 8, mapping the development in the understanding of the density of rational numbers.[2] To explicitly tackle the idea of discreteness, they implemented the *rubber line* metaphor (Vamvakoussi, Katsigiannis, & Vosniadou, 2009), focusing on the 'no successor' aspect of density, to help students transition through intermediate states of understanding, and showing different degrees of success depending on the students' grade level.

Another approach used by PME researchers to investigate notions related to functions and calculus is the development of models on the understanding of a given notion. Roh (2010a, 2010b) and Roh and Lee (2011) established a framework for understanding the ε-*N* definition of limit and constructed activities aimed at helping students grasp this definition, and, in particular, understand the role and order of the quantifiers ε and *N*. Their second activity (Roh & Lee, 2011) exploits the use of ε-*N* strips, and their results seem to indicate they help students develop a better understanding of the ε-*N* definition. The use of strips is not new, and in the early 80s, Robert (1983) showed the potential of such approaches through the use of didactical engineering, and, in particular, the effect of classroom interaction in the construction of the notions of limit and convergence. It is important to note that, although Vamvakoussi, Roh and their colleagues built cognitive models, a big part of their research is based on social interaction among students. However, this element is not central to their research and their analyses focus on the individual. We explore other approaches in Sections 3.4 and 4, where going beyond the individual becomes an important factor.

Finally, it bears mentioning that some works presented at PME conferences have used the Abstraction in Context (AiC) approach (Schwarz, Dreyfus, & Hershkowitz, 2009). This approach investigates processes of constructing knowledge by taking into account the need for a new construct as part of the process of abstraction. This approach follows three stages: (1) identifying the need for a new construct; (2) the emergence of the new construct; and (3) the consolidation of the new construct. This approach has been used, for instance, to investigate the process of constructing a definition – in this case, inflection point, which is a problematic object in calculus – (Gilboa, Kidron, & Dreyfus, 2013). AiC confirms once again that students often appeal to their concept image and not to the concept definition of mathematical objects, and demonstrates that students must realize the need for a definition. Also, Kouropatov and Dreyfus (2013) used the concept of accumulation to introduce the Fundamental Theorem of Calculus (FTC) – following the work of Thompson and Silverman (2008) – and showed how AiC is useful for identifying actions and constructs to study processes of knowledge construction. Their results point to the relevance of the notion of accumulation to help students construct processes of integration, and indicate that understanding of the FTC can be achieved based on the notions of accumulation and rate of change. Although the role of the interviewer seems to have a strong effect on results (and one might question the stability of the knowledge constructed by the students in their experiment), the effectiveness of identifying different levels of actions calls for more research using this approach. It is also worth noting that even though the focus of this approach is mostly cognitive, it has recently been linked with more social approaches, as we discuss in Section 4.3 on the networking of theories.

3. SEMIOTIC REPRESENTATIONS IN THE CLASSROOM: THE DEVELOPMENT OF THE CONCEPT OF FUNCTION

3.1. Use of Representations, Patterns and Variation

Over the last ten years, a large number of PME researchers have focused on the idea of generalization related to variation in primary school; this allows the subsequent use of this same approach in secondary school, employing patterns as a tool. In this approach, children in primary and secondary school are asked to find a general rule for a given pattern and to produce a semiotic representation to explain their reasoning. Through these processes, pupils eventually develop a way to figure out the form of a general term related to the pattern (usually, from a visual, natural language or numerical point of view, although in some cases, particularly in secondary education, an algebraic expression is asked). Using this approach and favoring whole class discussion in primary school, Dooley (2009) proposes to analyse patterns not just focusing on a single variable, but rather on the functional relationship between variables, showing that generalization and justification are closely aligned. Making use of a whole class discussion, it is legitimate to ask whether every single pupil retains the knowledge constructed in class. In her study with 10-year-old children, Warren (2006) identified different types of performances regarding patterns (p. 380):

1. No response;
2. Nonsense response;
3. Quantification of the growing rule in symbols (e.g. +3);
4. Quantification using specific examples (e.g. $2 \times 3 + 2$, $3 \times 3 + 2$);
5. Correct symbolic relationship using unknowns (e.g. $3 \times ? + 2$).

These categories clearly reflect that not all children are able to immediately develop processes of generalization. Related to this, Wilkie (2015) analyzed 102 7th-graders' performances in generalization activities with patterns, finding that 18.6% of the population used correspondence, 14.7% gave a rule using letters and only 2.9% expressed their results as an equation. These research findings seem to indicate that several steps must be taken to help young students successfully perform this type of activity. This agrees, on the one hand, with Radford (2010, 2011) who posits that engaging in early algebra thinking is not immediate or spontaneous, and highlights the idea that while early algebra could be promoted, it must take into consideration specific pedagogical conditions. On the other hand, Wilkie's findings also agree with Trigueros and Ursini (2008), who indicate that several steps must be taken to acquire the notion of variable as *unknown,* as a *general number,* or as a *functional relationship*. As we discuss in the next section, these steps are a necessary precursor to acquiring the concept of function.

3.2. The Use of Covariation between Variables, Modelling and Task-Design

The late 90s and the first decade of this century saw a flurry of research that produced important results concerning the construction of the concepts of covariation and function. The PME and PME-NA groups on representations studied the processes of visualization and conversion between representations, both with students and pre-service teachers (see, for instance, the *Journal of Mathematical Behavior* special issue, edited by Goldin and Janvier, 1998). The construction of the concept of function appeared to be more complex than expected with respect to students (Janvier, 1998) as well as high school teachers (Hitt, 1998). Both cases highlighted the importance of building the concept of function through conversion processes among representations (Duval, 1995). The discussions continued in PME-NA, which led Carlson (2002) to present a more accurate approach showing the importance of the subconcept of covariation between variables as a prelude to the concept of function, which is concretized with the Five Mental Actions of the Covariation Framework (p. 65):

- MA1. Coordinating one variable with changes in the other;
- MA2. Coordinating the direction of change of one variable with changes in the other;
- MA3. Coordinating the amount of change;
- MA4. Coordinating the average rate of change;
- MA5. Coordinating the instantaneous rate of change of the function.

This framework underscored the importance of modelling in the construction of the concept of function. For instance, Thompson (2008) addressed a major problem in learning mathematics related to the notion of meaning (as a coherent conceptual approach), and exemplified this notion with three contents: trigonometry, linear and, exponential functions. With respect to linear functions, he explained the importance of meaning in associating the notions of linear functions to rate of change, proportionality, and average speed, showing the value of modelling (see also Thompson and Carlson, in press). Furthermore, Musgrave and Thompson (2014) explored teachers' mathematical meanings as influenced by function notation, finding that teachers read function notation by stressing only the content to the right of the equal sign while neglecting the importance of covariation between variables. The ability to shift from a variational to a covariational type of reasoning seems to be far from evident, and some efforts have been made to help students with this process. Here we can cite the recent contribution of Johnson (2015), who developed activities to promote this shift among 9th grade students, using a dynamic computer applet and a task-design approach. The use of dynamic computer representations seems to be an interesting approach to help students grasp covariational relations.

Over the last few years, modelling in the learning of mathematics has taken on greater importance from primary to university levels (e.g. Blum, Galbraith, Henn, & Niss, 2007), and as said above, its use can help students grasp content related to functions. This is the case of the work presented by González-Martín, Hitt and Morasse (2008), which assigns importance to representations and modelling processes using a task-design approach and collaborative learning in a sociocultural setting. Their work shows that secondary students' thinking processes during modelling activities can promote covariational thinking about variables, allowing the notions of independent and dependent variable to emerge naturally. In their study, they introduced the notion of spontaneous representations (non-institutional representations) constructed by the students to tackle modelling activities, showing that these representations act as an important 'bridge' between the students' first attempts at tackling the activity and the institutional representations expected by the teacher and the school system. The work on modelling can provide a suitable environment to facilitate the evolution of these spontaneous representations in a special socio-cultural setting, called ACODESA (see also Hitt & González-Martín, 2015).

3.3. Transition from Mental Images to a Focus on Semiotic Representations and Visualization as a Semiotic Process Related to Functions and Calculus

As we said in the previous section, the PME working group on representations (Goldin & Janvier, 1998) and further research published by PME-NA (see Hitt, 2002) revealed another side of the learning coin, contrasting with the *concept image – concept definition* approach. In this perspective, external representations of mathematical objects are fundamental, because they permit the apprehension of mathematical concepts. Conversion processes between different representations therefore play an essential role in the construction of mathematical concepts, articulation among registers of representations becoming an essential part of the learning process (Duval, 1995, 1999). It is worth noting that targeted external representations are usually connected to pre-existing institutional representations, such as those found in textbooks or on computer screens, and shared by a collectivity. All the papers in this section clearly illustrate that this theoretical approach distances itself from the *concept image – concept definition* approach. The main goal is to understand the difficulties students experience when doing a treatment in the same register of representations, when converting from one representation in one register to another representation in a different register, and, of course, to know more about their learning and understanding. One of the main characteristics of this approach is that it relates visualization to a cognitive activity that is intrinsically semiotic (Duval, 1999; Presmeg, 2006a). Although its first versions placed a clear emphasis on the cognitive aspects of learning – and, as a consequence, on the individual – this led to a noticeable shift from investigation focused on mental images (or related constructs) constructed by an individual, to investigation focused on the conversion processes between representations, and, finally, on how the individual in a learning process constructs a mathematical concept throughout this activity (Duval, 2006; Presmeg, 2006b, 2008). Under this theoretical approach a new era of research emerged, exposing the learning problems that materialize when converting from one representation to another.

We discuss two specific examples concerning derivatives. To contribute to the debate on the differences and complementarity between visualization and analytic thinking, Aspinwall, Haciomeroglu and Presmeg (2008) constructed an instrument to better understand the thinking of calculus students, particularly with respect to derivatives. In their work, which aligns with PME contributions on visualization (Presmeg, 2006b), mathematical visualization encompasses "processes of creating or changing visual mental images, a characterization that includes the construction and interpretation of graphs" (p. 98). The instrument they constructed predicts individuals' preferences for visual or analytic thinking, showing that successful students use a combination of visualization and analysis, and that verbal-descriptive thinking helps sustain the use of visual and analytic thinking. Moreover, their work shows that visual and analytical processes are mutually dependent during mathematical problem solving, and that the verbal-descriptive component acts as a necessary link, being one of the most useful modes of internal processing, supporting visual and analytic processes.

Finally, Hähkiöniemi's (2008) research used aspects of embodied cognition (see Section 2.3) to investigate the meaningfulness and durability of students' knowledge. This study is related to the promotion of an articulation between the definition of the derivative of real functions and graphical representations of the function and the tangent of the curve in one point. The formal definition of the derivative was not addressed; rather, the study investigated descriptions of five 12th grade students who were assigned tasks of conversion between the definition of the derivative and a graphical representation of the situation (qualitatively analyzing the rate of change of functions from graphs), one year after receiving instruction. Regarding the meaning of the derivative, all the students referred to the slope of the tangent, the rate of change, and the differentiation, giving embodied meaning to the derivative and using gestures to describe it. The author suggests "it seems that the graphical and embodied elements of the derivative were experientially real for the students and gave meaning to the abstract mathematical concept" (p. 116). The visualization of the tangent also seemed to be a helpful tool for the students, assisting with the durability of knowledge and leading in many cases to the use of gestures.

These two last examples illustrate the potential of research on visualization and representations to make connections with other ways of expressing mathematics that are usually neglected by traditional practices: language and gesture. We come back to the potential of these approaches in Section 6.3.

3.4. Sociocultural Approaches to Teaching and Learning Covariation between Variables and Functions

As we discuss in Section 4, the last few years have seen the emergence of some (and the consolidation of many) institutional and sociocultural approaches in mathematics education. These approaches have also figured in work on semiotics and representations. The PME community quickly realized the importance of representations for the teaching and learning of mathematics, evidenced by the "Semiotics" discussion group organized by Sáenz-Ludlow and Presmeg held from 2001 to 2004. The work carried out in this group bore fruit, leading to a special issue of *Educational Studies in Mathematics* in 2006 (Sáenz-Ludlow & Presmeg, 2006) and the book, "Semiotics in mathematics education" edited by Radford, Schubring and Seeger (2008). In the introduction to this book, the authors point out that the theoretical approach of semiotics attempts to understand "the mathematical processes of thinking, symbolizing and communicating" (p. vii), adding:

> But semiotics is more than a contemplative gesture: in contemporary semiotic perspectives the notions of culture and cultural praxis receive a new interpretation—interpretation which extends to history as well—making semiotics a form of practical understanding and social action (Thibault, 1991). This is why it does not come as a surprise that semiotics is increasingly considered as a powerful research field capable of shedding some light on what have traditionally been understood as self-contained domains of enquiry. (p. vii)

This perspective represented a break with the cognitive approach. Research studies did not focus exclusively on the individual, and communication processes were ascribed more power. The use of semiotics in the processes of signification (in a construction of the sign and concepts) has been in ascendancy as a theoretical approach in PME over the last ten years, and in these processes, communication is the main ingredient in a sociocultural setting, as illustrated by González-Martín et al. (2008), discussed in Section 3.2.

Communication and its combination with artefacts (considered broadly and not restricted to technology) are paramount in the theory of semiotic mediation (TSM). Mariotti (2012), in considering a collaborative setting where communication is a main element, argues that:

> The theoretical model of TSM offers a powerful frame for describing the use of an artefact in a teaching-learning context. Within this model the use of an artefact has a twofold nature: on the one hand it is directly used by the students as a means to accomplish a task; on the other hand it is indirectly used by the teacher as a means to achieve specific educational goals. (p. 36)

From this perspective, the planning of teaching activities where communication is important and the use of processes of co-construction of the sign, including artefacts, requires a careful task-design. This can allow teachers to reach their goals while giving students the opportunity to construct, in this process of signification, action schemas through which the artefact evolves into a tool. The conceptual and technological approach to learning mathematics was influenced by semiotics, communication, and the transformation of an artefact into a tool, leading to a better understanding of the technological approach to the teaching and learning of functions (e.g. Presmeg, 2008) and calculus (e.g. Lagrange & Artigue, 2009) in a technological environment, as we will see in the next subsection.

3.5. Semiotics and Technology, the Concept of Function and Modelling Processes

As there is a chapter in this *Handbook* concerning the use of technology, we will mention just a few works produced in the last ten years that reflect the rapid evolution of research on the problems of learning functions and calculus in a technological environment. At the end of the last century, advances in technology led many countries to adopt a high school syllabus that promoted the teaching of functions and calculus using three representations – numeric, graphic and algebraic (e.g. Schwarz, Dreyfus, & Brukheimer, 1990) – and some conversion activities appeared in calculus textbooks. However, conversion is not easy, even when using technology. Following the theory of reification (Sfard & Linchevski, 1994) for the case of functions, Campos, Guisti and Nogueira de Lima (2008) showed how secondary school teachers could not shift from the interiorization and condensation phases to the reification phase in a computational environment, when confronted with tasks about conversion among representations.

Such studies illustrated once again that conversion among representations (from a cognitive perspective) is not as easy as was previously thought, even using technology. A shift was made, not only with respect to the cognitive approach using technology, but also in studying the role of communication in the process of knowledge construction. This shift led to new studies on the processes of instrumentation and instrumentalisation when dealing with artefacts and semiotic mediation (Arzarello & Paola, 2008; Hegedus & Moreno-Armella, 2008; Mariotti, 2012), promoting a better understanding of the concepts of variation, covariation and function.

With regard to modelling processes, technology reinforced pupils' possibilities to reflect on covariation between variables and the construction of functions in a dynamic approach; for example, using MathWorlds (e.g. Rojano & Perrusquía, 2007) and video-clips (e.g. Naftaliev & Yerushalmy, 2009). These environments allow the use of different technological perspectives, thereby providing more opportunities to implement modelling processes than in the past (see also Arzarello, Robutti, & Carante, 2015).

As discussed in Section 3.2, modelling processes are gaining importance in research, and technology expands the possibilities for new approaches in the classroom. The connections between modelling and technology were addressed in PME39, during the plenary lecture given by English (2015). This lecture presented the STEM (Science, technology, engineering, and mathematics) project related to the unification of several scientific branches to tackle common goals; it also discussed the importance of a STEM perspective in education and how this project is changing syllabuses and curricula in countries such as the USA and Australia. The fact that curricula need to emphasize more data analysis from modelling processes and functional approaches was stressed, as well as variation and covariation processes; the use of technology may pave the way for major advances in these areas.

4. OTHER APPROACHES

As mentioned in the introduction, Harel et al. (2006), in their discussion on future research, noted the predominance of cognitive approaches in research, stating that "It would be enlightening to incorporate social and cultural constructs […] offered by PME scholars, into AMT studies" (p. 162). Section 3.4 outlined how these constructs have been exploited through the use of semiotic approaches. In fact, the emergence and consolidation of such approaches has now affected all levels of education, their impact at the tertiary level being more recent, as shown in a recent *Research in Mathematics Education* special issue in which Nardi, Biza et al. (2014) stated: "we see the emergence of institutional, sociocultural and discursive approaches to research in [University Mathematics Education] as a milestone" (p. 91). In this section, we discuss some of these approaches and how they have enriched our understanding of the processes of teaching and learning of functions and calculus.

4.1. Institutional Approaches

In this section, we consider the contribution of the Anthropological Theory of Didactics (ATD, Chevallard, 1999) to the study of processes related to the teaching and learning of calculus. The use of ATD has developed quickly in recent years and has shown its potential to deepen the study of processes of teaching and learning from an institutional point of view. Its use at the tertiary level has grown considerably (Winsløw, Barquero, De Vleeschouwer, & Hardi, 2014) and the number of CERME conference participants applying it at the university level is considerable (Nardi, Biza et al., 2014). Paradoxically, its presence in PME regarding the teaching and learning of functions and calculus is still scarce.

Like other approaches in this section, ATD puts forward the view that mathematical objects are not absolute but emerge from human practices. A fundamental notion is that of *institution*, which is broadly defined as a social organization that allows and imposes on its *subjects* (every person who occupies any of the possible positions within the institution), the development of *ways of doing and of thinking proper to itself* (Chevallard, 1989, pp. 213–214). Therefore, regarding mathematical objects, institutions develop sets of rules that define what it means to 'know' these objects, thus determining their *institutional relationship* with mathematical objects, i.e., the *ideal* relationship that their subjects should have regarding these objects. Subjects also have a *personal relationship* with any object, as a product of all the interactions they can have with these objects through contact with them as they are presented in different institutions. Institutional relationships have a strong effect on personal relationships, and the study of learning processes requires an examination of institutional practices.

This effect is illustrated in González-Martín's (2013, 2014) research. This study on how pre-university textbooks introduce infinite series of real numbers (González-Martín, Nardi & Biza, 2011), also examined teachers' practices, identifying several implicit *contract rules* that could potentially influence students in their learning:

- To solve given questions about series, the latter's definition is not necessary.
- Applications of series, inside or outside of mathematics, are not important.
- The notion of convergence can be reduced to the application of convergence criteria.
- To solve given questions about series, the use of visualization (or any visual representation of series) is not necessary.

The results of this study, which examined a group of 32 pre-university students studying under three different teachers, showed the effect of institutional organizations (in this case, through textbooks) and teachers' practices on their students' learning. If existing *praxeologies* take for granted that visualization is developed in a natural, spontaneous way, and if they address issues related to convergence and its meaning solely through the application of convergence criteria, it is no surprise that students do not develop any tools to tackle questions requiring the development of visual abilities, nor do they develop an interpretation of what convergence really is, calling instead on intuitions or using the potential infinity (see Section 2.2).

ATD has also been used in PME, as well as outside PME, to investigate the transition from secondary to tertiary studies, highlighting the impact of institutional choices (many of which are guided by societal choices) on how content is organized and what students can learn. For instance, Alves Dias, Artigue, Jahn and Campos (2010) investigated the kind of tasks associated with functions in the selective evaluations that serve as gateways to the tertiary level in Brazil and France. Their analysis shows that in these evaluations, functions belong to different *habitats*: algebra in Brazil, and analysis in France. The type of tasks that put functions into play in both contexts is considerably different, thereby leading students to develop different skills and, consequently, different *personal relationships* with functions. Given the presence of ATD on the international scene and its potential to illuminate the effects of institutional choices (at several levels) on students' learning, its negligible impact on the PME community is surprising. It is also worth noting that this theory is connected to the Theory of Didactical Situations and instrumental and documentational approaches, which have also had little impact in the PME community over the last ten years. However, these approaches and their combination with ATD have proven useful for studying teaching and learning phenomena concerning calculus, both at the secondary and the tertiary levels (see González-Martín, Bloch, Durand-Guerrier, & Maschietto, 2014; Gueudet, Buteau, Mesa, & Misfeldt, 2014; Winsløw et al., 2014).

4.2. Commognition

The commognitive framework, which emerged in recent years and consolidated with the publication of Sfard's book (2008),[3] appeared very early in PME. This framework stresses the close relationship between thinking and communicating, to the point that "Thinking is an individualized version of (interpersonal) communicating" (p. 81), and sees learning as a change in ways of communicating. It identifies four distinctive features of mathematical discourses, analyzing how they change over time: word use, visual mediators, routines, and narratives. To illustrate this principle of learning, which can be seen as a change in discourse, we mention the work of Kim, Sfard and Ferrini-Mundy (2005), who analyzed students' discourse concerning infinity and limits, going beyond other works focused on misconceptions and cognitive obstacles. They investigated two groups of students as they aged (Korean and US students), comparing the characteristics and evolution of their discourse concerning limits and infinity. Their results show that the fact the word *infinite* appears in the English language before it is used mathematically – which is not the case in Korean – seems to influence the way students define and refer to infinity; this appeared to lead the US students to take the object-like character of infinity for granted, and the researchers concluded that colloquial discourse effectively seems to have an impact on mathematical discourse.

The commognitive approach was also used by Güçler (2011) to analyze the historical development of limits, identifying junctures that resulted in changes in the discourse on limits, and which may also be critical for students' learning. She highlights that the dynamic view, which holds an underlying assumption of continuous motion, dominated mathematicians' discourse until the 18th century, and that it was not until Cauchy (1789–1857) that the notion of limit was objectified. Although he realized the necessity of a theory of limits and an explicit definition of the concept, his definition still called for the metaphor of continuous motion. Weierstrass (1815–1897) and Dedekind (1831–1916) replaced Cauchy's kinematic approach with an algebraic-arithmetic approach: the metaphor of continuous motion was replaced with the metaphor of discreteness. Objectification led to the elimination of dynamic motion; however, in spite of the precision that the current formal definition of limits provides, it "wipes out all the intuitive tools with which to make sense of the concept" (p. 470). Data coming from a study with students (see also Güçler, 2013) seems to indicate that these junctures are critical for students: even when writing expressions such as $\lim_{x \to 4} f(x) = 2$, students will say "it is approaching two", rather than "the limit is equal to two". This word choice is seen as an indicator of students only endorsing the narrative 'limit is a process' and not objectifying limits as a number at the end of their instruction (p. 447). The metaphor of continuous motion is also present in some students, indicating that they did not attend to the instructor's shifts in word use and metarules in the contexts of the informal and formal definition.

This approach provides a lens through which to examine aspects of interactions in detail, and has also been used to study phenomena related to the transition from school to university mathematics (e.g. Nardi, Ryve, Stadler, & Viirman, 2014). Recent developments have adapted it to study teachers' knowledge in terms of discourse (Cooper, 2014) and this approach offers great potential to study "'the macro-level' of historically established mathematical discourse, the meso-level of local discourse practices jointly established by the teacher and students […] and the micro-level of individual students' developing mathematical discourses" (Cobb, 2009, cited by Nardi, Ryve et al., 2014, p. 196).

4.3. Networking of Theories

The comparison of theories has been a subject of interest for the PME community. For instance, at the beginning of the century, Boero et al. (2002) worked on comparing theories of abstraction and, in the last ten years, Presmeg (2006a) compared two theoretical frameworks: that of Duval (1995, 1999) on semiosis and noesis (related to the articulation among registers of representations) and the semiotic means of objectification of Radford (2002, 2003).

The work of Prediger, Bikner-Ahsbahs and Arzarello (2008), and later of Bikner-Ahsbahs and Prediger (2009), which examined the importance of focusing on the networking of theories, may have spurred the PME scientific committees' promotion of research along these lines (with two Research Forums in 2010 and 2014, see the introduction of this chapter) in order to unite, differentiate and strengthen different theoretical frameworks. According to Clark-Wilson et al. (2014), the aim of networking theories is to unite, differentiate and strengthen different theoretical frameworks to better explain learning phenomena. This brings us to the discussion undertaken by the PME forum organized by Bikner-Ahsbahs et al. (2010), who identified some conditions for an efficient networking: "the underlying principles have to be 'near enough' and […] the empirical load of a concept plays a crucial role if integrating is the aim" (p. 146). Furthermore, they offered a broad analysis of different ways in which theories can be networked (Figure 1):

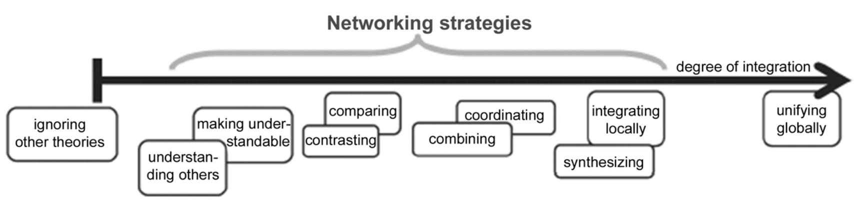

Figure 1. Networking strategies (Prediger et al., 2008, p. 170)

One of these ways is comparing and contrasting. Taking this into consideration, it is possible to make a critique of Presmeg's (2006a) discussion when comparing relationships amongst signs related to the theoretical approaches of Duval (1995, 1999) and Radford (2002, 2003). As explained in Section 3.3, Duval's approach mainly concerns institutional representations and the conversion processes among them that serve students in the construction of an articulation among representations related to a mathematical object. Meanwhile, according to Radford (see Section 3.4), in the process of objectification, elements are considered that are not necessarily institutional, but are an integrant part of the institution, such as culture, communication and representations. That is, the manipulation of objects, drawings, gestures, marks, and the use of linguistic categories, analogies, metaphors, etc., are key components of mathematical communication in the process of objectification. Presmeg's comparison analyzes excerpts from both theoretical approaches, albeit in a context related to institutional representations (concerning trigonometric functions, graphic representations, and processes of visualization). It is therefore not surprising that Duval's approach is seen as "paramount" (p. 32), but we can criticize the fact that the analyzed data are not 'near enough' Radford's theoretical approach, and the design of the investigation seems rather to have been conceived to be analyzed from a cognitive approach and not from a semiotic process of signification. This contrasts with the principles of the networking of theories, as highlighted by Bikner-Ahsbahs et al. (2010), in which research must be designed to allow the integration of different approaches, giving both theories the chance to interact (Figure 1).

One example of networking of theories, implying the use of technology, is provided by Clark-Wilson et al. (2014). They contrast theoretical approaches and constructs that have been frequently used to examine students' performances and have recently been applied to teachers – for instance, *instrumental genesis* and *instrumental orchestration*. New constructs emerged from this contrast to analyze students' errors or strategies when solving problems (not restricted to functions and calculus), such as *critical incidents*, *hiccups*, and the notions of *instrumental distance* and *double instrumental genesis*. A similar evolution took place with the notion of *Pedagogical Content Knowledge* (PCK, Shulman, 1986), which led to the emergence of the *Pedagogical Technology Knowledge* (PTK, Thomas & Hong, 2005); the latter focuses on mathematics and employs the theoretical base of instrumental genesis.

Finally, we mention the work of Kidron, Bikner-Ahsbahs, Cramer, Dreyfus and Gilboa (2010), who networked the Interest-Dense Situations (IDS) and the Abstraction in Context (AiC, see Section 2.4) approaches in activities concerning real numbers to assist in "uncovering blind spots of their methodologies" (p. 169): IDS considers social interactions as a basis for learning mathematics, and AiC develops tools to investigate the construction of learning, with social interaction as part of the context. Their paper shows how each theory helped uncover or refine elements of the data analysis performed by the team using the other theory, noting that, for instance, "the social interaction analysis offered by IDS reveals important cognitive aspects" (p. 175).

4.4. Mathematics Education, Psychology and Neurosciences

We end this section by looking at the increasing importance placed on interdisciplinary research between mathematics education and psychology and neurosciences at the last PME meeting (PME39), which leads to a questioning of "how different methodologies currently used in cognitive neuroscience afford, and constrain, research design and potential findings/implications for maths education" (Tzur & Leikin, 2015, p. 115). We name two works that illustrate this interdisciplinarity in recent research in mathematics education. Lithner (2015) presented a descriptive study to show how the brain works when solving tasks related to an algorithmic reasoning and when solving tasks related to creative mathematically founded reasoning; among the results reported, brain imaging seems to indicate that students learning by creative reasoning could use their mental resources more economically in different tasks. This work also relates to that of Waisman, Leikin and Leikin (2015), who used tasks related to functions and proposed to measure "mathematical ability" through the identification of brain activity. Their results also show that different mathematical abilities reflect in different ways on ERPs, and that these differences are dependent on the level of insight imbedded in the task solution. These works introduce new dimensions for research in mathematics education and in some cases seem to confirm or contradict some of the beliefs held about students' learning. Although the presence of functions and calculus in the examples briefly presented here is peripheral, we believe that new areas can be explored to improve our understanding of how this content is learned.

5. TOPICS DESERVING SPECIAL ATTENTION

In this section we discuss topics that have been the focus of growing research in recent years, especially concerning functions and calculus. Space limitations have forced us to zero in on just a few topics. This obliges us not to discuss, for instance, studies that have been developed in the last few years on the secondary-tertiary transition, particularly with different approaches;[4] one example of these studies is given in Section 4.1.

5.1. Teachers' Knowledge and Practice

The theoretical proposition of Shulman (1986), concerning the amalgam of specific content knowledge and teaching knowledge, gave origin to what we know as PCK and to a field of research on teachers' knowledge. Research now focuses on preservice teachers to a greater extent than in the past, particularly regarding content related to functions and calculus. In this sense, research has identified that a strong PCK concerning a mathematical topic (inverse functions, in the case of Bayazit & Gray, 2006) does not guarantee adequate teaching. This result calls for further research to identify factors orientating PCK towards effective practices, avoiding procedural approaches (such as the ones identified by Lucus, 2006, concerning the composition of functions).

Developing Shulman's (1986) work for the teaching and learning of mathematics, Ball and Bass (2000) introduced a new dimension for teachers' knowledge, the *Mathematical Knowledge for Teaching* (MKT). Using this construct, Seago and Goldsmith (2006) worked with teachers possessing a conventional and 'compressed knowledge' of linear functions; they showed that some activities demanding conceptual and 'unpacked understanding' presented in a seminar can help teachers increase their MKT and develop their teaching abilities. However, the use of professional development materials does not always produce a significant impact on teachers' MKT, as Seago, Carroll, Hanson and Schneider (2014) discuss in the case of linear functions. The MKT construct has also been used to explore lower secondary mathematics teachers' abilities concerning mathematical language (Wang, Hsieh, & Schmidt, 2012). Results unveil difficulties teachers may have concerning competences related to thinking and reasoning about mathematical language, as well as difficulties they may have choosing teaching activities that could cultivate their students' competences related to mathematical language.

PME has also been interested in teachers' beliefs (see the chapter *Research on mathematics-related affect* in this Handbook). However, research on calculus teachers' belief systems is still scarce, and work in this area may make a useful contribution to an emerging area of calculus research (Rasmussen, Marrongelle et al., 2014, p. 512). In Section 5.3, we refer to some of these works concerning calculus in tertiary education. Regarding calculus in high school, Erens and Eichler (2014) were interested in the structure of belief systems, which characterize teachers' instructional planning. Working from the perspective of beliefs and goals, their work identifies some relations of coordination (for instance, presenting calculus as process-oriented and application-oriented) and subordination (for instance, presenting calculus as application-oriented as a means to facilitate students' motivation). However, their work also identifies contradictions that may be due to personal factors (for instance, although they hold a formalist view of calculus, some teachers do not activate it to avoid creating difficulties for students) or to constraints from external factors (for instance, although an instrumentalist view could be peripheral for a teacher, s/he could activate this goal in order to help students pass national exams). These results support the idea that beliefs and goals depend not only on individual factors (as we also illustrate in Sections 5.3 and 5.5 concerning tertiary education), and identify the existence of inconsistencies, calling for further research. Other inconsistencies, this time regarding the use of visualization, have been identified by Biza, Nardi and Zachariades (2008): some teachers can explicitly accept a visual argument, while at the same time claiming the need to support and verify algebraically for the same statement. The possibility of holding erroneous images, combined with a tendency to support visual reasoning in some cases, could lead to negative effects from the use of visualization.

5.2. Teachers and Task-Design

Task-design is not new, and already at the beginning of the twentieth century, psychology studies on intelligence had developed specific tasks suitable for these studies (Brownell, 1942). The evolution of different theoretical frameworks in mathematics education entailed the production of tasks associated with these frameworks, the coherence between the task and the theoretical approach being a fundamental element of research design. For instance, considering the notion of epistemological obstacle (Brousseau, 1997) related to the concept of function, and to raise awareness of the conception that "functions are continuous and expressed by a single algebraic expression", one task that has proven efficient consists of asking participants to construct two examples of one real variable function such that for all $x, f(f(x)) = 1$ (Hitt, 1994).

In reflecting on ways to improve task-design activities and the efficacy of these tasks, the 2014 Research Forum on mathematical tasks (Clark, Strømskag, Johnson, Bikner-Ahsbahs, & Gardner, 2014) was structured around four key questions:

1. What are the possible functions of a mathematical task in different instructional settings and how do these functions prescribe the nature of student task participation?
2. What contingencies affect the effectiveness of a mathematical task as a tool for promoting student higher order thinking skills?
3. How might we best theorize and research the learning processes and outcomes arising from the instructional use of any mathematical task or sequence of tasks from the perspective of the student?
4. What differences exist in the degree of agency accorded to students in the completion of different mathematical tasks and with what consequences? (p. 119–120).

These questions can provide a suitable basis for the discussion on the elaboration of efficient tasks, taking into account the mathematical content, the student and the teacher. Furthermore, a reflection on and transformation of the tasks used may have the potential to change teaching and learning approaches to functions and calculus, as highlighted, for instance, by English (2015).

Finally, regarding the use of technology, in a work related to functions and modelling processes and following the documentational approach, Psycharis and Kalogeria (2013) highlighted the existence of three factors that may hinder teachers' construction of tasks and materials: (1) Teachers' difficulties in developing their own teaching material, (2) Teachers' difficulties in learning the affordances of the software tools and integrating them into activities with added educational value (e.g. microworlds, scenarios, worksheets), (3) Teachers' knowledge, pedagogical conceptions and experiences regarding the everyday practice of teachers. These results highlight the need for additional research on task-design activities in technological environments (Clark-Wilson et al., 2014), particularly regarding functions, calculus and modelling.

5.3. Teachers, Teaching Practices and Their Effects in Tertiary Education

In the last PME *Handbook*, Harel et al. (2006) acknowledged the growing body of research on mathematicians' writing, problem solving, and proving (p. 160). Although they also mentioned the growing research on mathematicians' teaching practices, there was not enough space in their chapter to develop this point. More recently, with respect to research on the teaching and learning of calculus, Rasmussen, Marrongelle et al. (2014) identified work on teacher knowledge, beliefs, and practices as one of the most recent developments in the field. Furthermore, as we noted in the introduction, covering the topics of functions and calculus in a single chapter would lead us to encounter some gaps. In this section, we address an important one. PME researchers have focused a good deal on teachers' beliefs, as the last PME *Handbook* shows (Leder & Forgasz, 2006). However, although primary and secondary teachers' training, beliefs and practices have been a subject of research for many years, this is not the case for the tertiary level, and there is still little research on how lecturers actually teach at the university level (Speer, Smith, & Horvath, 2010; Weber, 2004). In this section, we give an overview of some important results obtained in PME regarding these issues.

Regarding undergraduate teaching practices relative to calculus content, we cite the works of Rowland (2009) and Petropoulou, Potari and Zachariades (2011), related to university teachers' training, beliefs, decisions, and practices, the former in connection to the Fundamental Theorem of Calculus and the latter in connection to general calculus content, with data from sequences. In both cases, the authors developed a single-case study concerning a particular subject: a lecturer with a background in mathematics and mathematics education. In both cases, different uses of examples are among the main strategies aimed at constructing mathematical meaning. These works indicate that the instructors' practices appear to be based on the professional knowledge they develop (or craft), their beliefs and vision concerning the nature of mathematics itself, the purposes of teaching and learning mathematics, and the ways in which mathematics is most effectively taught and learned, as well as their own experience. These results agree with those of Weber (2004), who noted that beliefs about mathematics as a research mathematician, and beliefs about students and teaching as an experienced mathematics lecturer, were the main influences on a lecturer's practice. We see here differences in research concerning primary and secondary teachers, who usually receive teacher training that influences their belief system. This shortcoming in the case of tertiary education may place higher importance on the need to hold discussions within teams of lecturers, as proposed by Rowland (2009). This is justified by the fact that if an instructor's beliefs are his/her own, and if other lecturers teach differently and do not articulate similar beliefs, it is unclear which version of 'being mathematical' students might construct. Rowland therefore recommends that the entire team of lecturers meet to discuss different ways of teaching, epistemological assumptions, students' role in lectures, and so on, in order to establish sociomathematical norms.

Having examined university teachers' training, beliefs and practices regarding calculus, it is worthwhile to question the effects of these elements on students. So far, experience and the literature suggest that there is much research to conduct, because Calculus courses prompt many students to change careers: this issue is addressed by Rasmussen and Ellis (2013) who sought to better characterize the profile of students who choose not to continue with Calculus and uncover the main reasons why students switch out of Calculus courses. Their data comes from an in-depth national survey with over 14,000 students responding to at least one of their instruments. One important result showed that 12.5% of STEM-intending students in their sample had planned to take Calculus II at the beginning of their Calculus I course, but decided not to do so upon completing the first course. This group, called switchers, displays a number of characteristics: the percentage of female switchers is significantly higher in comparison to males (20% and 11%), switcher rates differ significantly depending on career choice (engineers having the lowest rate), and, the mathematical background of switchers and students who go on to Calculus II was statistically similar at the start of their post-secondary education. This last result is quite significant and refutes a preconceived idea: their data indicate that students who abandon their STEM ambitions are not weaker when they enter university than those who continue on the STEM path. Regarding the reasons for changing majors, 31.4% of students in this situation acknowledged that their experience with Calculus I made them decide not to take Calculus II. Their study also points to teaching practices as influencing students' experiences and choices, and indeed, the researchers' subsequent paper (Rasmussen, Ellis, Zazkis, & Bressoud, 2014) shows that one characteristic of successful calculus programs is the existence of substantive graduate teaching assistant (GTA) training programs, varying from "a weeklong training prior to the semester together with follow up work during the semester to a semester course taken prior to teaching" (p. 37). This training of GTAs, who are seen as future lecturers, was the topic of a paper by Ellis (2014), who sees it as a way to make up for the lack of pedagogical or didactical training for university teachers. The need pointed out by Rowland to hold group discussions on practices, views and beliefs can be addressed somewhat preemptively through professional development programs. Ellis (2014, p. 13) underlines the important role played by mentoring with respect to K-12 professional development, and her results seem to show that mentoring augments the training of GTAs, suggesting "a relationship between GTA professional development and student success that needs to be further examined" (p. 14). Given the large number of Calculus students around the world, research concerning teachers' practices and training is needed, as "there is great need to better understand the factors that contribute to student decisions to stay in or to leave a STEM major" (Rasmussen, Marrongelle et al., 2014, p. 512).

5.4. Analysis of Textbooks

Sträßer (2009) addressed the important role of artefacts (such as textbooks, computers, tools, etc.) in the teaching and learning of mathematics, acknowledging that textbooks "have always played a major role in mathematics education" (p. 70). This being the case, it is surprising that research has not placed much focus on the analysis of textbooks until recently, and research concerning high school and university topics – including calculus – is still scarce.

Of all the topics included under the label 'Calculus', a wide variety have received little research attention, especially the most advanced topics. However, of the topics that students encounter first, continuity is one that has been studied the least by researchers, often appearing implicitly in studies on limits or functions. Taking this into account, Giraldo, González-Martín and Santos (2009) analyzed how continuity of single-valued real functions of one real variable is presented in undergraduate textbooks used in pre-service mathematics teachers' calculus courses. Their main results indicate that the notion of continuity is mostly introduced using the notion of limit, in many cases using intuitive images that call for the image of 'a curve drawn without removing the pencil from the paper.' This could have consequences for teachers' understanding of the notion of continuity, which has already been signaled as problematic (Hitt, 1994; Mastorides & Zachariades, 2004).

Regarding the concept of infinite series, Nardi, Biza and González-Martín (2009) analyzed a set of university textbooks used in the UK (and which are also used in many other countries). The analysis focused mainly on the use of visual representations, tasks, and examples to introduce series, finding that this concept is mostly introduced in a decontextualized way, with few graphical representations and even fewer applications and references to the concept's significance and relevance. The results agree with the analysis of a larger sample of pre-university textbooks used in Quebec (González-Martín, Nardi et al., 2011). In addition, the effects of these textbooks and their use by teachers on students' learning of series have been analyzed using ATD (see Section 4.1). Finally, regarding secondary education, González-Martín, Giraldo and Souto (2011, 2013), analyzed how real and irrational numbers are introduced by textbooks, using ATD. Their results revealed a similar situation, as well as a lack of justification for the need of these 'new' numbers. Moreover, although studies have identified difficulties in learning the topics addressed in this section, this research is often neglected, which raises the question of why research on calculus is not having a greater impact on practices and resources.

5.5. Calculus as Service Mathematics

One of the areas of tertiary mathematics research that have developed rather quickly in the last few years is the study of educational processes for audiences enrolled in faculties other than mathematics (Artigue et al., 2007). The landscape has changed a great deal: technology has altered the skills and knowledge required for many of these professions, the number of students enrolled in these faculties has dramatically increased, the background (particularly concerning mathematics) of students entering these faculties has changed, and societal expectations have also grown. The number of papers published in the last few years on the teaching and learning of mathematics as a service course has increased, and the field of engineering has attracted special attention. However, efforts are still needed to better understand the phenomena at play, and the claim made by Kent and Noss (2001, p. 395) 15 years ago seems to be still valid, namely that "The teaching of service mathematics remains relatively unexplored, and many of its fundamental assumptions (What is its purpose? What are the fundamental objects and relationships of study?) remain unexamined."

Earlier in this chapter (Section 2.1), we mentioned the work by Bingolbali and Monaghan (2008), indicating the differences in students' acquisition of the derivative according to their department of affiliation. This work is closely related to their paper presented at PME30 (Bingolbali et al., 2006) where they analyzed the views and practices held by lecturers teaching Calculus courses in different departments. This research collected data from six different lecturers with experience teaching mathematics or physics as a service subject. Their results indicate that lecturers behave in different ways according to their audience: they privilege different aspects of mathematics, place different questions on examinations, and use different textbooks. For instance, lecturers emphasized different aspects of topics based on the type of student: concerning derivatives, aspects related to rate of change were highlighted for engineering students, whereas aspects related to tangents were highlighted for mathematics students. The role and place given to proof also varied according to the audience. But not every decision is the result of personal choice, and the lecturers' perception of the department's priorities also seems to play an important role. The results of this research have at least two implications. First, the connections between this work and the results presented by Bingolbali and Monaghan (2008) seem to imply that students' learning is strongly conditioned by their lecturers' choices. This aligns with many results obtained using ATD, such as those already noted by González-Martín (2013, 2014) in Section 4.1. Secondly, lecturers' choices are influenced by the fact that while they each have their own background, they see themselves as members of an institution (department or faculty), although these institution-driven choices can sometimes conflict with their own background and views, as illustrated by Hernandes Gomes and González-Martín (2015).

This type of investigation calls for more research to better understand the interplay of elements in contexts such as the teaching of calculus to engineering students. For instance, the choice of textbooks – and resources in general – may have implications for students' learning, as well as the training of the lecturers, as acknowledged by research following the documentational approach (Gueudet et al., 2014; Gueudet & Trouche, 2009). Also, while some choices seem to be made to adapt lectures to a specific audience, the question remains whether these changes lead to a 'different' calculus course, or whether the same key characteristics endure. In this sense, Barquero, Bosch and Gascón (2011) identified a *dominant epistemology* in university teaching that has an impact on different mathematics teaching practices. They call this epistemology 'applicationism' and its main characteristics are: (1) mathematics is independent of other disciplines; (2) basic mathematical tools are common to all scientists; (3) the organization of mathematics content follows the logic of mathematical models instead of being built up by considering modelling problems that arise in different disciplines; (4) applications always come after basic mathematical training; (5) extra-mathematical systems could be taught without any reference to mathematical models (pp. 1940–1941). Whether these principles can be found (and to what extent) in the practices of lecturers who 'adapt' content to their students' profile also remains an open question for research.

6. FUTURE RESEARCH

We finish this chapter by examining issues that, in our opinion, warrant further research. Once more, we focus on just a few issues, although we are aware that several require more investigation (we have noted some of these in previous sections of this chapter).

One important issue already identified by researchers (Artigue, 2001; Rasmussen, Marrongelle et al., 2014), is the fact that while research in calculus has concentrated on a few topics (namely functions, limits, derivatives, and integrals), advanced topics remain relatively unexplored. For instance, the proportion of papers focusing on differential equations is quite small compared with functions and derivatives, and papers on multivariate calculus are few in number. Issues concerning transition in a broad sense (for instance, from high school to university, from calculus to analysis, from calculus to algebra, etc.) also deserve further research through a variety of lenses, especially institutional and sociocultural perspectives. And, as we noted in Section 5.5, there is a great need to investigate the relationships between calculus and client disciplines in terms of practices, what should be taught, and what students are learning, to cite just a few. In particular, "Post-secondary educational research has from this point of view a specific epistemological role to play in educational research thanks to its proximity with the professional world of mathematics. The increasing importance taken in post-secondary mathematics education by service courses faces us with the necessity of taking a wider perspective" (Artigue et al., 2007, p. 1044). This leads us, finally, to underline the importance of coordinating efforts to make various advances in research concerning functions and calculus available to the broader practitioner and policymaking communities. Furthermore, systematic research on teaching practices concerning calculus content, particularly at the tertiary level, is needed.

In the following paragraphs, we address other issues that warrant further research.

6.1. Networking of Theories

As noted in Section 4.3, for more than 15 years the PME community has been interested in the comparison and/or networking of theories, and theoretical advances are shown in this line under different perspectives. Bikner-Ahsbahs et al. (2010) offer some methodological elements to consider in this process of networking and they draw on research projects, such as TELMA and Re-Math, identifying the emergence of cross-experimentation methodology as a key element. As noted by Bikner-Ahsbahs and Prediger (2009), to overcome some of the limitations that arise from using only psychological approaches, the networking of theories appears to be a promising way of doing research. Furthermore, considering technology (and, in particular, its use by client courses), Rasmussen, Marrongelle, et al. (2014) also propose the networking of different theoretical perspectives and their respective findings as a promising way forward. Although PME has considered the networking of theories, more systematic research is needed. As we have highlighted throughout this chapter, many issues related to the teaching and learning of functions and calculus interact (teachers' training or the lack thereof, practices, beliefs, materials and resources, departments, etc.), and the networking of theories looks to be a promising way of taking into account several of these issues at the same time.

6.2. Task-Design

Rasmussen, Marrongelle et al. (2014) recently stated that "It is noteworthy that the research in calculus learning and teaching has not capitalized on advances in design research […] to further link theories of learning with theories of instructional design" (p. 509). Although not necessarily connected with calculus, task-design has been the focus of some interest recently, as indicated by the organization of topic study groups focusing on it at ICME conferences, resulting in an ICMI study on task-design (Watson & Ohtani, 2015). This interest can be explained, according to Clark-Wilson et al. (2014), by the great difficulty teachers face in building tasks and applying them in the classroom. One strategy for successful task-design consists of proposing sequences of enchained tasks covering broad mathematical topics (Artigue, 2002), preferably aiming at producing emergent models, necessary to symbolize and mathematize gradually (Gravemeijer, 2007). As discussed in Section 3.2, Hitt and González-Martín (2015) proposed ways (a method) to tackle these issues in the classroom at the secondary level: in a sociocultural approach, the construction of a sequence of activities promoting diversified thinking and the emergence of non-institutional representations. Combining individual work, teamwork and whole class debate, these tasks helped pupils co-construct the subconcept of covariation between variables, necessary to the construction of functions. This is just one example of task-design, but certainly more effort must be made to transfer results on students' learning of calculus into design research. At the undergraduate level, collaboration between mathematicians and mathematics education researchers seems to be a promising avenue for future research.

Finally, concerning technology, there is also a need (both for pre-university and university students) to relate theoretical work with computer activities, as well as the need to create sequences of activities around a given topic.

6.3. Semiotics

In looking at research developed over the last 10 years in the PME community concerning argumentative discourse, two types of components can be highlighted:

- A component that seeks to convince, to win the support of the other (called the seduce component by some authors).
- A component that aims to explain, based on reasoning.

These two components can be found in different works of the PME community: the first is seen more in research on collaborative learning when communicating mathematical ideas in primary and secondary school (more related to conjecturing and convincing in peer interaction), and the second is found mostly in research on university-level contexts, where proof is a requirement even if a previous conjecture has been made. Both components are always present in research on communication with others, but in the construction of mathematical thinking, instruction usually promotes the gradual diminution of the first component and an incremental increase in the second one. Some authors following a semiotic approach include gestures in the argumentative discourse – as we discussed in Section 3.3, – which is an interesting approach. As Sfard (2008, p. 94) puts it: "while defining thinking as individualized communication, I was careful to stress that all forms of communication need to be considered, not just verbal."

Many results related to semiotics do not adequately clarify the relationship between the process of resolution of the mathematical task (not exclusively with mathematical content) and the role the students or teachers assign to gestures to convince others. The importance of spoken language is not always highlighted either, as we discussed in Section 3.3. We believe that Sections 2.3 and 3.4 have emphasized the important role that gestures and signs (other than mathematical symbols) can play in the learning of mathematics, and certainly more research is needed to better understand how semiotics, in a sociocultural setting, can help in the understanding of learning processes concerning generalization, functions, calculus and modelling problems in context.

NOTES

[1] For a summarized overview of the Theory of Didactic Situations and the main principles of didactical engineering, particularly at the undergraduate level, see González-Martín, Bloch, Durand-Guerrier & Maschietto (2014).

[2] These authors also studied students' self-confidence, showing that students who gave better answers were also usually more confident in their answers, whereas this was not the case when ends were given in the form of decimal numbers (0.8 and 1.1 in their case). Issues surrounding affection and self-confidence are discussed in the chapter *Research on mathematics-related affect* in this Handbook.

[3] For a summarized overview of the main tenets of this framework, and some examples of its use at the undergraduate level, particularly in calculus, see Nardi, Ryve, Stadler & Viirman (2014).

[4] For instance, Rasmussen, Marrongelle et al. (2014) state that "the secondary vs. tertiary differences might be greater when viewed through a pedagogical or cultural lens, including institutional constraints and affordances. This is an interesting area of research" (p. 507).

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