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Students' perceptions of the completeness property of the set of real numbers

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This article presents an exploratory study that gives account of some students' perceptions of the completeness property of the set of real numbers. Students taking three undergraduate correlative courses in Calculus and Analysis answered a written questionnaire; their scripts are analysed trying to understand how operational is the notion of completeness for them and to what extent they take this notion for granted.

Keywords: real numbers; completeness; university level; students' perceptions

1. Introduction

The set of real numbers **R** is the domain conventionally used in Calculus and in the first courses in Analysis taught at the university level. However, we can hypothesize that first-year students neither have a clear understanding of what this domain represents nor what the properties of **R**, on which the work in Calculus and Analysis is founded, are.

In most universities, the set of real numbers is approached in a progressive manner throughout several courses. In the first Calculus courses the set **R** is not explicitly defined; instead, what is used is the naive idea of considering **R** as all possible numbers, influenced by the image of the number line. This idea is compatible with the kind of work usually carried out in these courses, and it allows instructors to advance through the curriculum fairly quickly. Further on, when studies in Analysis start and the formally defined set **R** becomes the natural domain of functions, other properties become relevant. **R** is then usually introduced by means of the axioms of a complete ordered field. In advanced courses, **R** is presented as the set of rational cuts or rational Cauchy sequences. What do students understand and misunderstand about **R** and completeness along that path?

The purpose of this article is to offer an account of undergraduate students' mathematical images or perceptions of **R** and its completeness, via the analysis of the answers given by students of three undergraduate correlative courses in Calculus and Analysis. The set of real numbers and its defined property of completeness is a topic that can be studied from a didactic point of view on a long-term basis, since it is taught, at various depths, in several undergraduate courses. Above all, it is a topic that is at the core of Analysis, and reflects some general principles of this field, like

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determining an element by means of a nested sequence of sets, or defining one element as the limit of others under a certain condition, or achieving an extreme value, among others.

2. Background

The essential difference between **R** and the other ordered numerical fields is given by the property of completeness. This property can be expressed by several equivalent characterizations, some of them being:

- Every non-empty and upper-bounded subset of **R** has a least upper bound (supremum) that belongs to **R**.
- Every bounded sequence of elements of **R** has a subsequence convergent in **R**.
- Every Cauchy sequence is convergent in **R**.

The notion of completeness was developed at the end of the nineteenth century in order to allow a reliance on a numerical system in which all finite or infinite decimal representations represented a well-defined number; and also in order to rely on a numerical system with the attributes that had been naturally assigned to the points of the line: order, density and continuity. In this way, it was possible to prove some Calculus theorems that could not be proven before on an arithmetical basis only [1].

The introduction of the concept of completeness becomes necessary when we need *to prove* (instead of *to take for granted*) the existence of certain numbers such as maximum, minimum and some limits, leaving aside the idea of natural continuity.

To the best of my knowledge there are few works in research in mathematics education that explicitly focus on the concepts of completeness of **R** and continuity of the line. Castela [2] claims that the correspondence numbers-points is used in teaching, in the hope to transfer to the set of numbers the properties intuitively assigned to geometrical objects. Nevertheless, not only does this correspondence implicitly entail that the line is provided with density and continuity, but it also assumes that the correspondence numbers-points itself is not problematic for the students: two conditions which are not necessarily fulfilled, as was showed in Castela (op.cit.). Nuñez *et al.* [3,4] refer in their works to two conceptions of the number-line. One of these conceptions is spontaneous and intuitive; the line is conceived as a global entity, naturally continuous, where the points lie *on*. For the other conception, the line consists of the union of (uncountable) points regarded atomically. Both lines are cognitively different and consequently for students they provide different inferential structures.

There are several other works in mathematics education concerning real numbers which are mostly focused on teachers' and students' conceptions about rational and irrational numbers, such as the works of Robinet [5], Barbosa and Da Silva [6], Fischbein *et al.* [7], Sirotic and Zazkis [8], Bronner [9] to mention a few. The mathematical aspects that are considered in these works are the decimal expansions, the representation of numbers on the line, density, cardinality and the necessity to introduce numbers other than those that are rational. Beyond the diversity of their theoretical and methodological frames, there are some points of consensus among them. First, students at different scholastic levels hold inconsistent conceptualization regarding rationality, irrationality, discrete order, dense order and continuity. Second, representing numbers on the line does not necessarily help to reduce those difficulties, as all students do not conceptualize the line in the same way, and consequently referring to its perceived attributes does not bring a secure support. These works, most of which address learning and teaching at the secondary level or at the end of that cycle, do not focus on completeness but they constituted for me a valuable departure point for analysing students' ideas on **R** when its properties become essential to offering proof and theoretical justifications.

3. A survey concerning completeness: goals and methodology

My goal was to explore what students think **R** is, what representations they have about it and to what extent they overcome the idea of natural continuity through different courses in Analysis. Drawing on notions developed in the Theory of Conceptual Fields by Vergnaud [10], a central position that I adopted is that a concept cannot be reduced to its definition, but it acquires its meaning through problems and situations. Accordingly, my initial questions were of the sort: Is the statement 'a real bounded above non decreasing sequence has a limit' taken by students as a manifestation of completeness or as evidence that it is a powerful tool to show convergence? When students learn that 'a continuous function in a bounded closed interval attains the maximum', or when they learn the intermediate value theorem, do they relate these results with the completeness of the domain or do they use them as evident theorems assuming completeness as something pre-existent or rather pre-constructed (in the sense of Chevallard [11])? I assume that an analysis of personal conceptions cannot be done in the isolation of what students have learned, consequently I observed the opportunities students had to work with completeness through four courses of undergraduate studies in mathematics at the University of Buenos Aires (in what follows, I will name them by Courses I, II, III and IV) by analysing the tasks students had to do, the techniques they used and their technical and theoretical justifications [12]. Regarding these tasks, in Course I students are demanded to find maxima, minima, least upper bounds (sup) and greatest lower bounds (inf) of some subsets of **R**, which is done based on the graphic representations of the subsets in question. In Course II, students have to find maxima, minima, sup and inf of more complex sets and to justify their answers by means of the definitions. I interpret this requirement of justification as a change in the didactic contract, where graphic representations are no longer accepted as a full basis for justifying an answer. Students do not understand the reasons of this change as no new tasks are proposed to support this difference of perspective. In Course III sup and inf appear in a more instrumental role (in defining distances for instance), and also as objects that admit more than one definition, for which the equivalence students are asked to prove. In Course IV, students have to prove the equivalence of five different ways of defining completeness.

I carried out two inquiries. The first one consisted of a written questionnaire that was answered by the majority of the students in Courses II, III and IV (124 out of 192 students in Course II, 11 out of 24 in Course III and 10 out of 16 of Course IV). The second one consisted of interviews of volunteer students from the three courses. In this article, I present a part of the written questionnaire.

4. The written questionnaire

The written questionnaire consisted of the following five questions:

(1) If you wished to explain to a younger student that a real non-decreasing bounded-above sequence has a limit, how would you do it?
(2) What do you think the notion of least upper bound (supremum) is useful for?
(3) What does '**R** is a complete set' mean for you? (only for Courses III and IV)
(4) Considering what you learned in Analysis up to now, what 'parts' require the use of the axiom of completeness? In other words, what concepts and properties of numbers and functions would not be possible to know without this axiom?
(5) For each $c \in \mathbf{Q}$, consider the function f_c: $\mathbf{Q} \to \mathbf{Q}$ given by $f_c(x) = x^2 - c$. (Even though it is not defined in **R**, it is a continuous function.)

   (a) Is there any value $c \in \mathbf{Q}$ such that the following statement be true: If f_c takes two values $f_c(a)$ and $f_c(b)$ then it takes all the values between them?'
   (b) Would something be different if f_c were defined in **R** instead of **Q**?

In this article, I will present my results concerning Questions 1 and 3. I used the students' answers to the first question to see to what extent completeness is problematized for the students. Do they question themselves about the existence of a limit? Do they assume its existence as something evident or take it for granted? For instance, if they translate the statement to a graphic frame, the existence of the limit is no longer problematical: when students see in a drawing the sequence accumulating, it is likely that for them the existence of a limit becomes evident. It is important to examine if the drawing is regarded as constituting the complete answer or offers a representation allowing an explanation. Another thing to observe in the answers is to what extent completeness of **R** appears as a necessary condition, as the statement *a non-decreasing bounded-above sequence has a limit* is not true unless the sequence is a real one.

For students in courses III and IV, who had already studied the property of completeness as a part of their syllabus and manipulated it on several occasions, I included the third question: *What does '**R** is a complete set' mean for you*? I included this question to observe if students gave their answer in terms of properties and conditions or mostly in terms of images of natural continuity. For the first case, where completeness can be used in exercises and problems, we may say it is *operational*.[1]

Note that both questions are phrased in such a way to encourage expositional answers. This is done deliberately to dissuade the overt usage of technical terms that might in actuality not be fully comprehended by the student.

For each course, after reading all the student scripts, I grouped the answers in types in order to analyse the 'degree of problematization' of completeness for the first question and to examine the extent for which the idea of completeness is operational for students for the second question. I identified five types for the first question and two main types for the second one.

In the next paragraphs I will present the students' answers of the first question for the three relevant courses, and then the answers to the second question for courses III and IV. Representative examples of each type of answer are included.

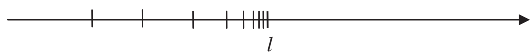
5. Question 1: main types of answer

Five types of answers were identified to the question: *If you wished to explain to a younger student that a real non-decreasing bounded-above sequence has a limit, how would you do it*?:

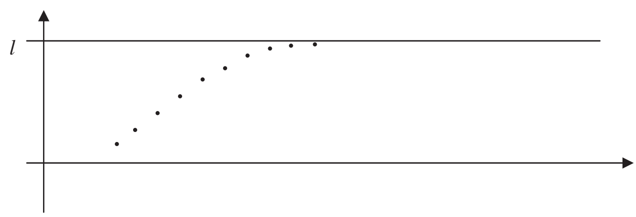
Type 1: Students justify the existence of the limit, mentioning the property of completeness in any of its forms.
Type 2: The existence of the limit is not put in question, but taken for granted.
Type 3: Some students give a metaphorical answer, in an extra-mathematical context.
Type 4: For some students the sequence necessarily stabilizes, that is, it is constant after a certain sub index.
Type 5: Unclear answers, or no answer at all.

Some students included one of the following two drawings in their answers:

Drawing 1:



Drawing 2:



5.1. *Type 1*

Students justify the existence of the limit. Some of them mention the existence of the least upper bound; other students say that there is a theorem or an axiom that states that. Some examples of answers Type 1 are:

‘Let be $(a_n)_n \in {}_N$ a real sequence. $\exists\, c \in \mathbf{R}/|a_n| \le c \forall n \in \mathrm{N}$. $a_n \le a_m \forall n, m \in \mathrm{N} \wedge n < m$. By axiom of completeness, there exists a least upper bound c' such that $a_n \le c' \forall n \in \mathrm{N}$

$$\Rightarrow c \text{ is the lim it of } (a_n)_{n\in N} \text{ as } \forall \varepsilon > 0\ \exists\, \nu = \nu(\varepsilon) \text{ such that } c' - \varepsilon/2 \le a_\nu \le c' \Rightarrow$$
$$c' - \varepsilon/2 \le a_\nu \le a_n \le c' \forall n > \nu \Rightarrow -\varepsilon/2 \le a_n - c' \le 0 \forall n > \nu,\ \Rightarrow 0 \le c' - a_n \le \varepsilon/2 \Rightarrow$$
$$\left|c' - a_n\right| = \left|a_n - c'\right| < \varepsilon.\text{'}$$

‘I should explain that the sequence is non-decreasing and bounded above, which means that it is always lower than a real number. Therefore, as it grows and it is lower than a number the terms of the sequence will approach a number l (l may be different from the upper bound, not all bounds are limits, the limit is the unique which is the best upper bound, the supremum).’

‘If the student has not done Course II, I would explain that non-decreasing and bounded above sequences always have a limit. And that is true only if the sequences are defined in R, otherwise it is not true. For instance, $a_n = 1 - 1/n$ is non-decreasing and bounded above, and therefore it must have a limit, the limit is 1. But also we can have a sequence like F_n, such that $F_{n+2}=F_{n+1}+F_n$, $F_2=F_1=1$. It diverges, but $a_n=F_n/F_{n+1}$ verifies that $a_n \subseteq \mathbf{Q}$, $a_n \leq 1$, $a_{n+1} \geq a_n \forall n \in N$. However, this sequence only converges in the set of real numbers, as its limit is $(\sqrt{5}-1)/2$, which does not belong to rational numbers. It does not converge to a value in **Q**.'

I grouped these questions in a type considering students' recognizing of the role or need of the completeness' axiom in an application that requires it.

5.2. *Type 2*

The existence of the limit is not put in question. Students explain the meaning of each and all terms of the statement. They explain what the notions of sequence, non-decreasing sequence, bounded-above sequence and convergent sequence mean. At a certain moment they write something to the effect: 'though, it necessarily tends to the limit' or 'intuitively, it must converge' or 'then, it has no option but to converge'. Some examples of this type of answer are:

'A sequence $(a_n)_{n\in N}$ is non-decreasing $\Leftrightarrow \forall n_0 < n_1,\ a_{n_0} < a_{n_1}$. A sequence is bounded $\Leftrightarrow \forall a_n \in (a_n)_{n\in N}\ |a_n| < M$. Consequently, the terms of the sequence are bigger and bigger and at the same time are lower than a number L, that implies $\exists \lim_{n\to\infty} a_n = L \Leftrightarrow \forall \varepsilon > 0\ \exists\ n_0/\forall n \geq n_0$: $|a_n - \mathrm{L}| < \varepsilon$' (Drawing 1). 'The terms of the sequence are near and nearer M (an upper bound of $(a_n)_{n\in N}$).'

'I would firstly explain what a sequence is and what it means that it is non-decreasing and bounded above. From that, it is easy to see what the limit represents, it is a value to which the sequence approaches as I want, but I never touch it' [Drawing 2].

'As we can see in the drawing, the sequence is non-decreasing – as it takes values bigger and bigger each time. At the same time, it is bounded above, that is, all the terms are lower than a number, M in this case. The sequence never takes the value M, then I can say that from a certain value, all the points of my sequence are nearer and nearer M. I can say that my sequence tends to M.'

'First, if a sequence is non-decreasing, then from $n=1$ the values it takes will be bigger each time $a_1 < a_{n+1} < a_{n+2}$. It is bounded above, that is there exists M in **R** such that all a_n will not be bigger than M. This implies: $\exists M \in$ R such that $\forall n \geq n_1$ $|a_n| < |a_{n+1}| < |a_{n+2}| \cdots < M \quad \lim_{n\to\infty} a_n = l$ (finite number) and $l < M$.'

'First, I would show a drawing (Drawing 1). If a sequence is non-decreasing, it is due to the fact that the numbers are bigger at each time. Bounded is that there is a kind of wall the numbers will not pass. Numbers will accumulate near the wall, but they will never touch the wall.'

'I would use a drawing (Drawing 2). As the sequence is non-decreasing, it goes up, as it is bounded above, never can surpass a top. If we have a bounded sequence, it cannot diverge to ∞. Then the unique form that it does not converge is that it oscillates (like sinus, cosinus). If we add the hypothesis of non-decreasing, it cannot oscillate, therefore, it has a limit.'

'A sequence has three possibilities: it converges, it diverges, it oscillates. If it converges, that is it. If it diverges: $\forall n \in \mathrm{N}\ \exists a_n \geq n$, that is, the sequence is not bounded. If the sequence oscillates, it cannot be, as it is non-decreasing.'

The argument of the two last answers is that a sequence converges, diverges or oscillates (something that is true only if the sequence is defined in **R**). Completeness is passed by: its role is implicit, as a property something that naturally holds.

5.3. *Type 3*

Some students give a metaphorical answer, in an extra-mathematical context; e.g. by comparing a non-decreasing and bounded-above sequence with the steps of a stair, or drops in a glass. Some examples of this type of answer are:

'I would suggest to think that you are climbing, your mountain is bounded as it has a top, to reach the top is to approach the limit, therefore, your mountain has a limit. The limit will be the highest point that is reached.'

In this answer the limit is interpreted as a maximum.

'What is a sequence? Idea: it is an ordered set of things. Non-decreasing: each term is lower than the following. Idea of non-decreasing and bounded above: imagine pupils in a school, sorted by their height.

a b c d

$a<b<c<d$. Height of pupils has a maximum, the height d. d is an upper bound, the highest height pupils can have.'

In this answer the situation is viewed as discrete, moreover, as finite, and the limit is obviously given by the maximum.

'It is easier if we think of an example. We think the sequence as a staircase. If the sequence is non-decreasing, we can say that each step is higher than the former one. If the sequence is bounded above, we say that the staircase reaches to a closed door. I can approach it but I cannot surpass it, as I reached the limit of the staircase, that is, the door.'

This answer shows another context where the sequence is viewed as having a finite numbers of terms.

5.4. *Type 4*

For 2 students (out of 124) the sequence necessarily stabilizes, that is, it is constant after a certain sub index. An example of this type of answer is:

'A sequence can be thought as a function from **N** to **R**. If it moves only in a way, it is non-decreasing. It arrives to a point that is a top, it does not matter to what value the function is applied, it image will always be that point.'

Here, students think the sequence as a discrete set.

5.5. *Type 5*

Eight students gave unclear answers, and one student did not answer.

5.6. *Synthesis of the answers given to Question 1*

In my view, for almost all students in Course II, the existence of the limit is taken as being evident or transparent. For the majority of them (98 out of 124) it is so transparent that it is sufficient for each and all terms of the statement to be explained, the exception being the existence of the limit, which seems to require no explanation! Other students (10 out of 124) consider an extra-mathematical context, where the word *limit* is taken in the context of something of daily use and it exists without any questioning. Through an explanation or a drawing students stress – without making it explicit – the fact that the sequence is a Cauchy sequence, and the limit 'is there' (as one can 'see' in Drawing 1). Fifty-two out of 53 drawings go together with a Type 2 answer (Section 5.2). I interpret this major tendency as an evidence of a pre-constructed and natural vision of completeness. However, it must be taken into account that the wording of this question (to explain to a younger student) may favour the tendency of producing answers Types 2 and 3.

In the majority of the answers given by the students of Courses II and III, the existence of the limit is not put in question. These answers are somehow out of phase with the exercises and problems that the students are supposed to have done in the course. The raising of images and non-mathematical terms (a man walking up to a wall, a barrier) is surprising, as they were never used in textbooks or courses. Such images constitute a support for some students, and belong to their private ideas. It is interesting to consider what kinds of problems would persuade them to put aside these images and make them realize the need to use more operational definitions (Tables 1 and 2).

6. Question 3

Answers to the question: *What does 'R is a complete set' mean for you?* by students in Course III and IV

I grouped the answers of this question in two main types:

Type 1: an operational definition, given by a mathematical expression of completeness
Type 2: a natural vision of completeness, somehow referring to the word 'complete' as full, there are no places, there are no gaps, or also mentioning the real-line and an intuitive notion of continuity.

Table 1. Types of answers given to Question 1 by students of Courses II, III and IV.

	Using completeness explicitly	Taking limit for granted	Metaphorical answers	Sequence stabilizes	Unclear or non answer
Course II	$5/124 \approx 4\%$	$98/124 \approx 79\%$	$10/124 \approx 8\%$	$2/124 \approx 2\%$	$9/124 \approx 7\%$
Course III	$5/11 \approx 46\%$	$4/11 \approx 36\%$	$2/11 \approx 18\%$	0%	0%
Course IV	$2/10 \approx 20\%$	$8/10 \approx 80\%$	0%	0%	0%

Table 2. Use of a drawing as a part of the answer to Question 1.

Course II	$53/124 \approx 43\%$
Course III	$3/11 \approx 27\%$
Course IV	$3/10 \approx 30\%$

6.1. *Answers Type 1*

Some answers are close to being operational, describing **R** as a set such that the cuts of numbers, in the sense of Dedekind, that are neither jumps nor lacunas[2] (3 out of 11): '**R** is complete is the same as **R** is continuous. That is, it has no jumps, nor lacunas. Between two points of **Q** there is one irrational, and between two irrational there is a point of **Q**. Then, there's no gap on **R**.'

'**R** is complete, that means it has no jumps, nor lacunas. It has the property of supremum, and it is continuous.'

Other operational answers: 'There are as many real numbers as points on the line, and upper bounded sets have a supremum', 'Every Cauchy sequence is convergent to an element of the set', 'It has no gaps, every bounded-above set has a supremum and every Cauchy sequence is convergent', 'Completeness guarantees the existence of several elements, via the axiom of supremum or other equivalent.'

6.2. *Answers Type 2*

Some answers evoke the everyday meaning of the word complete as replete, there are no 'empty places', there are no gaps. For instance: '**R** is complete, as it contains all the numbers', '**R** has no gaps'.

Some answers refer to the real-line: 'Real numbers cover the line, to each point corresponds a real number' 'real numbers complete the line, between two real numbers there is another, one cannot take the former number', 'Completeness is a line without gaps, an infinity density of points', '**R** is complete, you can see it if you represent **R** as a continuous line, every point is a real number. If we take **Q**, for instance, that does not happen, because between two rational numbers there is a real number. If I cut a line with another line, there is a real number there. That is, it has no gaps. I think of complete as replete, without empty spaces. An example could be the line, the line of Euclid, a sequence of infinite points, one glued to another . . .'.

The last sentence of the last answer shows that the representation on the line does not necessarily help students to better understand what completeness is, as in this case the line is viewed as a sequence of points (numerable).

Several students seem to have an intuitive idea of what completeness is, but this idea is not sufficient to be used in proof or in exercises. Still a substantial proportion of students recall the notion of line, but it does not necessarily help them to better conceptualize the notion of completeness (Table 3).

7. Synthesis and conclusions

In spite of the limits of the questionnaire, the answers show that only a few students of Course IV see completeness as a tool to define new elements. A possible

Table 3. Types of answers given to Question 3 by students of Courses III and IV.

	Operational answers	Natural vision of completeness
Course III	$1/11 \approx 9\%$	$10/11 \approx 91\%$
Course IV	$7/10 \approx 70\%$	$3/10 \approx 30\%$

explanation to this can be related to the fact that students can use advanced theorems that no longer explicitly refer to first principles, so that they are not obliged to directly face completeness. Most of the students do not know which problems completeness solve.

Considering the tool-object dialectic introduced by Douady [13], I conclude from the results that the 'object' aspect of completeness is surprisingly weak for students in Course III, given the type of tasks the students solve in this course. The majority of the students expressed completeness in a non-operational way: by means of the daily use of the word 'complete' or by means of images. In Course IV this changes: 7 students out of 10 give an operational characterization.

The expression 'complete means that it has no gaps' referring to completeness can be thought as a degenerate version of the statement 'every cut of the set has a unique element of separation'. In a similar way, the expression 'getting closer' that appeared in some answers is a way of talking about 'approximation', in the sense that the distance between two objects can be made less than any particular positive number. Both are examples of weak and non-operational images of mathematical definitions. It might be interesting to design learning situations to help students to turn these images into mathematical statements.

Understanding completeness as a property or an axiom that settles a critical mathematical issue requires a reflection that does not seem to appear spontaneously as a result of solving the given exercises. For most of the students, doing typical exercises involving the supremum does not lead to the understanding that **R** is the set that contains all the suprema of its bounded above subsets. Few students can perceive that the notion of *Cauchy sequences* come from the necessity of characterizing the kind of sequences that 'must' converge – an essential insight required to further develop mathematical analysis – and that completeness is related to the issue whether a limit is guaranteed to lie in **R**. All these aspects, that in general remain in the sphere of the private work of the student, are elements that I think are important to take into account when preparing syllabi and designing exercises.

Notes

1. As one of the reviewers suggested, it might be more accurate to denote this type of answer as *potentially operational* rather than *operational* (if so, it would be *actually operational* for advanced students and mathematicians that do operate with the definition).
2. Jumps and lacunas are kinds of cuts. For instance, all cuts in the set of integer numbers are lacunas, all cuts in the set of rational numbers are jumps.

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